Reduction of one and two loop Amplitudes at the Integrand level

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Reduction of one and two loop Amplitudes at the Integrand level

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Doctoral thesis

to obtain the degree of doctor
from Radboud University Nijmegen
on the authority of rector magnificus
prof. dr. S. C. J. J. Kortmann,
according to the decision of the Council of Deans
to be defended in public on Monday, 2 july 2012
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Chapter 1

Reduction at one loop

1.1 Introduction

We are living in a very interesting era for particle physics. Recently (30 September 2011) the Tevatron was shut down leaving an enormous amount of data to be analysed. At the same time, the Large Hadron Collider (LHC) is working very well challenging our understanding of particle physics with its experiments. In order to understand the output of these experiments, comparison between very precise theoretical results and experimental results is needed. It is clear, from the theoretical point of view, that Next-to-Leading-Order (NLO) calculations with many external legs have to be considered.

An important part of the NLO calculations is the loop calculations. When considering processes with many legs at one loop, one has to calculate large loop integrals, a procedure that for many years was considered to be the bottleneck of these calculations.

Reduction techniques form a way out. The idea of reducing Feynman integrals with a large number of denominators to a set of simpler integrals (simpler in our case means with less denominators) goes surprisingly many years back [1],[2]. A typical integral with n such denominators is given below

$$\int d^4q \frac{1}{D_1 D_2 ... D_n} \tag{1.1}$$

, where $D_i = (q + p_i)^2 - m_i^2$ is the inverse of the propagator.

In [2] the authors reduce a triangle (integral with 3 denominators) to bubbles (2 denominators) in 2 dimensions while in [1] a pentagon (5 denominators) is reduced to boxes (4 denominators) in 4 dimensions. We will be able to reproduce their results with a slightly different method later. We see that the result of the reduction depends on the number of spacetime dimensions. However, the methods we will use can be applied to all dimensions. Our main interest of course is 4 space-time dimensions.

What followed was the "renowned" paper of Passarino and Veltman [3]. With the use of Lorentz invariance the authors proved that every one-loop integral with a tensor structure in the numerator can be decomposed to scalar integrals with less or equal number of denominators in the renormalizable gauge. As a consequence, only the evaluation of scalar integrals (integrals with trivial numerators) is needed to perform a one-loop calculation. The method is still in use nowadays. Examples of a Passarino-Veltman decomposition will appear later in this text and then we can explain in more detail the method.

In [4] another pentagon to boxes decomposition is performed in 4 dimensions. The importance of this paper is that it provides a basis (the so called van Neerven-Vermaseren basis) very fruitful for understanding a lot of important results in one-loop reductions. Another important fact about this paper is the use of what we call nowadays spurious terms to decompose a scalar pentagon to boxes. Spurious terms are terms that vanish upon integration. Their role will be explained later when we will consider reductions at the integrand level. We will show explicitly how to construct them and how to use them for any one-loop reduction.

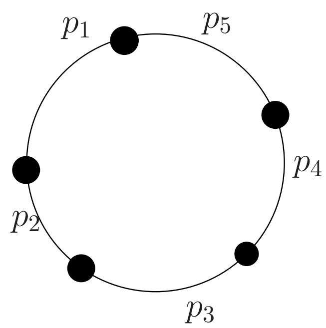
The next big step comes from on-shell methods [5]-[14]. Instead of working with specific Feynman diagrams these methods have a big advantage in that they try to decompose the whole amplitude. By cutting propagators (putting them on-shell) the rational coefficients of loop integrals are given in terms of products of tree amplitudes. In generalised unitarity methods, the notion of quadruple (triple and so on) cuts is introduced. One can cut more than one propagator (notice that in 4 dimensions cutting 4 propagators freezes the loop momentum) to find these coefficients.

The Ossola-Papadopoulos-Pittau (OPP) method [15] comes as a natural combination of all the above. Since every integral can be decomposed to scalar integrals up to 4 denominators (in 4 dimensions), every one-loop amplitude is written in terms of coefficients that multiply these scalar integrals. It works at the integrand level and that means that in order these decompositions to be possible one must also include spurious terms. Then one has to find a way to calculate the coefficients of the reduction and multiply them with the appropriate scalar integrals, using one of the numerous packages available for the evaluation of them (i.e [24],[25]). Finding the coefficients is a purely algebraic problem. The method is suitable for a fully numerical implementation. The OPP method will be extensively described in a next chapter since it is the main subject of this thesis.

1.2 Reduction with trivial coefficients

We start looking at the integrands of any given one-loop amplitude. These integrands consist of the sum of every integrand coming from the Feynman diagrams that contribute to a given process; the advantage is that we perform the decomposition once instead of reducing every single diagram separately. It is obvious that since all these integrands are lumped together in one big denominator there is no notion of momentum conservation along the loop[1]. For that reason we deal with integrand-graphs, or iGraphs instead of Feynman diagrams. We give an example of an iGraph of order 5 (pentagon) below, where

$$\begin{aligned} D_j \equiv D(q+p_j) = (q+p_j)^2 - {m_j}^2 = q^2 + 2(p_j \cdot q) + \mu_j \quad , \\ \mu_j \equiv {p_j}^2 - {m_j}^2 \quad , \end{aligned} \tag{1.2}$$



[1]The reason we don't assume momentum conservation is because we want to study properties of general integrands. We don't want to use properties or symmetries that hold on specific occasions and thus are not valid in a general case. In case we refer in an example to a specific Feynman diagram though, we will make it clear.

A one-loop pentagon iGraph. The dots serve to distinguish the several denominators.

The loop momentum is denoted by q^μ, and ${p_j}^\mu$ is called the *external momentum*, where it must be realized that by this we do *not* mean a momentum related to a particle incoming our outgoing in a given amplitude: what we call external momenta are simply fixed momenta, given in some way by the configuration of incoming and outgoing momenta and the various diagram topologies.

Consider a one-loop iGraph of order n:

$$\frac{1}{D_1 D_2 D_3 D_4 ... D_n} \quad .$$

We say that we can decompose this iGraph *if* we can find funtions $T_{1,2,\ldots,n}(q)$ such that

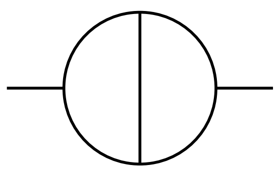
$$D_1 T_1(q) + D_2 T_2(q) + \cdots + D_n T_n(q) = 1 \quad , \tag{1.3}$$

for then we have

$$\frac{1}{D_1 \cdots D_n} = \frac{T_1(q)}{D_2 D_3 D_4 ... D_n} + \frac{T_2(q)}{D_1 D_3 D_4 ... D_n} + \cdots + \frac{T_n(q)}{D_1 D_2 D_3 ... D_{n-1}} \quad , \tag{1.4}$$

and the original iGraph is decomposed into a sum of iGraphs of order $n-1$ (or lower).

This immediately leads us to state the following theorem: *one-loop iGraphs of order d or smaller* **cannot** *be decomposed in the above manner.* The reason is simple: for $n \leq d$ there exist a cut through *all* propagators, so that $D_j = 0$ for $j = 1, \ldots, n$ and eq. (1.3) then would become $0 = 1$. Similarly, at L loops an iGraph of order dL or lower cannot be thus decomposed. This does *not* imply that iGraphs of higher order must always be decomposable. A counterexample is the *Feynman* diagram of order 5 in two dimensions:



If all internal lines in this self-energy Feynman diagram are massless, it is possible to choose the four loop momenta components such that all five propagators are simultaneously cut. Needless to say, this calls for special circumstances[2]. However, the above theorem does not shed much light on the *mechanism* by which the iGraph becomes non-decomposable, and in the following we shall still investigate these 'hopeless cases' as well.

The simplest possibility for the functions $T_j(q)$ is to take them to be just numbers independent of q^μ ('trivial' coefficients):

$$T_j(q) = x_j \quad .$$

From eq. (1.3) then we have

$$q^2 \sum_{j=1}^{n} x_j + 2q_\mu \sum_{j=1}^{n} x_j {p_j}^\mu + \sum_{j+1}^{n} x_j \mu_j = 1 \quad . \tag{1.5}$$

Since this has to hold for *any* value of q^μ we must have separately

$$\sum_{j=1}^{n} x_j = 0 \quad , \quad \sum_{j=1}^{n} x_j {p_j}^\mu = 0 \quad , \tag{1.6}$$

and

$$\sum_{j+1}^{n} x_j \mu_j = 1 \quad . \tag{1.7}$$

Note that if a nontrivial solution to the homogeneous equations (1.6) exists, then by suitable scaling we can always satisfy eq. (1.7). We see that, at one loop, for $d = 4$ any iGraph of order 6 or higher can be decomposed in this fairly trivial way. A pentagon in 4-dimensions thus cannot be decomposed that way. For general d, iGraphs of order $d + 2$ or higher are decomposable.

1.3 Reduction with linear coefficients

For a one-loop iGraph of order 5 (or lower) no trivial decomposition exists in $d = 4$. Indeed, by shifting the loop momentum we can always arrange to have $\sum_j {p_j}^\mu = 0$, so that for $n = 5$ the only solution to the 5 conditions in

[2]This diagram *can*, in fact, be decomposed, but not by the method described above: instead one has to use integration-by-parts techniques. We will deal with that case explicitely when we will consider two loop cases

eq. (1.6) is $x_j = 0, j = 1\dots5$ which is unacceptable in eq. (1.7). We therefore turn to the next simplest possibility for the T's, with a linear q dependence:

$$T_j(q) = x_j + \sum_{k=1}^{4} x_{j,k}(q \cdot t_k) \quad . \tag{1.8}$$

The single x_j is now replaced by 5 (or $d+1$) variables to be determined in each T. Here, the four (or d) vectors ${t_k}^\mu$ must be linearly independent but are otherwise arbitrary. In analogy to eq. (1.6) and eq. (1.7), we now have more tensor structures in terms of the loop momentum: we can denote them by the shorthand

$$1 \ , \ q^\mu \ , \ q^\mu q^\nu \ , \ q^2 q^\mu \ . \tag{1.9}$$

There are, for $d = 4$, therefore 1+4+10+4 = 19 independent tensor structures. Note that the q^2 appearing in eq. (1.6) is no longer independent since it appears as the trace part of $q^\mu q^\nu$. This can be extended to the inclusion of higher-rank tensors and other dimensions: in d dimensions, and with the inclusion of tensor up to rank k, we find for the number $N(d,k)$ of independent tensor structures

$$N(d,k) = \begin{pmatrix} d-1+k \\ k \end{pmatrix} + \sum_{p=0}^{k+1} \begin{pmatrix} d-1+p \\ p \end{pmatrix} \quad . \tag{1.10}$$

In the table below we give the results for various ranks and dimensionalities.

k	0	1	2	3	4
d =1	3	4	5	6	7
2	4	8	13	19	26
3	5	13	26	45	71
4	6	19	45	90	161
5	7	26	71	161	322
6	8	34	105	266	588

Values of $N(d,k)$

The number of coefficients x to be solved for is given by

$$X(n,d,k) = n \begin{pmatrix} d+k \\ k \end{pmatrix} \quad . \tag{1.11}$$

Since for $d = 4$ and $k = 1$ we have $N(4,1) = 19$, it would seem that iGraphs of order 5 and 4 are decomposable with linear terms. However, the situation is not so simple since it is not obvious that the 25 coefficients for $n = 5$ and the 20 coefficients for $n = 4$ allow us to actually build up the 19 required tensor structures. We now describe how we can ascertain the number of independent structures numerically, by an approach that may be dubbed *cancellation probing*.

We start by generating *random* values for the external momenta ${p_j}^\mu$ and m_j $(j = 1, \ldots, n)$. This avoids any possibility of us choosing, coincidentally, any special phase space point where degeneracy might occur. Then, we choose *random* values for q^μ precisely $\xi = X(n, d, k)$ times, and insert all this in eq. (1.3). We are left with a set of ξ linear equations for the ξ unknowns x:

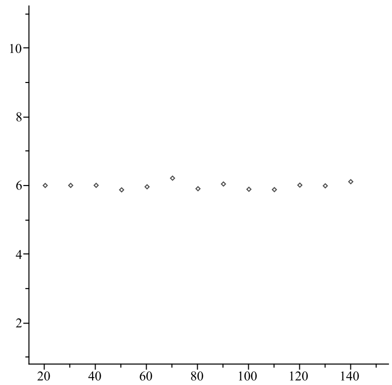
$$\sum_{j=1}^{\xi} {M^i}_j x^j = 1 \quad , \quad j = 1, \ldots, \xi \ . \tag{1.12}$$

The $\xi \times \xi$ matrix M is purely numerical. We obtain it using the computer-algebra package `MAPLE`[26] which, although not numerically the fastest available, has the essential advantage that one can set the precision with which numerical operations are performed[3]. Now, if the number of independent tensor structures that can be formed with our T's is less than ξ, the determinant of M will vanish. In an ideal real-number model of computation, we would thus find $det(M) = 0$, but in our actual numerical computation there will be rounding errors. A cancellation of numbers to 'zero' will , in `MAPLE`, actually give a number of order 10^{-p}, where p is the number of digits specifies in the precision we tell `MAPLE` to use. If the matrix' determinant is computed by Gaussian elimination[4], then a matrix with q zero eigenvalues will have a determinant of order 10^{-pq}. By letting p run down from 150 to 20 in steps of 10, we can obtain[5] a very accurate estimate of q, especially since q must be integer. We give two examples to demontrate the use of this method for the calculation of the zero eigenvalues. In both examples we consider a decomposition of pentagon to boxes, with linear and quartic terms respectively. We plot the outcome for different precisions in each case and we give also a table with the value of the determinant for each precision.

[3] The relevant variable is `Digits`.

[4] This is almost unavoidable since the matrix M is *not* sparse, and anyway we can choose Gaussian elimination as an option in any case.

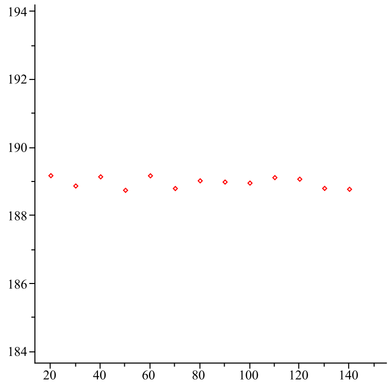
[5] Surprisingly, the cancellation probing appears to fail for $p = 15$ and $p = 10$, possibly since `MAPLE` may have special ways to treat these accuracies ($p = 10$ is default in `MAPLE`).



The number of zero eigenvalues using rounding errors for the case of a pentagon decomposition with coefficients linear in the loop momentum.
x-axis: precision digits, y-axis: number of dependent coefficients
size = 25, dependent = 5.987506357, "+/-", 0.09848735960

precisionDigits	*Value of the determinant*
140	1.601964842 10-826
130	1.014940068 10-765
120	- 1.017432149 10-706
110	3.576908545 10-647
100	- 4.483548004 10-587
90	1.531352890 10-526
80	-1.019160411 10-466
70	2.038084447 10-406
60	-3.313024775 10-346
50	- 2.059708869 10-287
40	9.759320292 10-228
30	5.854763845 10-166
20	- 2.354020369 10-108

Values of the determinant for different precision digits-linear coefficients



The number of zero eigenvalues using rounding errors for the case of a pentagon decomposition with up to quartic in the loop momentum coefficients.
x-axis: precision digits, y-axis: number of dependent coefficients
size = 350, dependent = 188.9831232, "+/-", 0.1529378959

$precisionDigits$	$Value\ of\ the\ determinant$
140	2.666045328 10-26000
130	3.534962869 10-24112
120	-2.389931149 10-22221
110	-4.641302331 10-20330
100	2.238171545 10-18440
90	-2.173038675 10-16550
80	4.879075288 10-14660
70	-5.975718074 10-12772
60	-3.957810308 10-10880
50	1.440683023 10-8992
40	4.779833159 10-7101
30	3.225707007 10-5212
20	-2.343227781 10-3320

Values of the determinant for different precision digits-quartic coefficients

The difference $\xi - q$ then gives the rank of M, and *this* determines the decomposability of the iGraph: the rank must at least equal $N(d,k)$ for it to be decomposable. In the table below we give the results of cancellation probing for various n and d.

n	$d=6$	$d=5$	$d=4$	$d=3$	$d=2$	$d=1$
2	14-0	12-0	10-0	8-0	6-0	4-0
3	21-1	18-1	15-1	12-1	9-1	6-2
4	28 -3	24-3	20-3	16-3	12-4	8-4
5	35-6	30-6	25-6	20-7	15-7	10-6
6	42-10	36-10	30-11	24-11	18-10	12-8
7	49-15	42-16	35-16	28-15	21-13	14-10
8	56-22	48-22	40-21	32-19	24-16	16-12

The rank of M for various d and n,
given as the difference $\xi - q$.

We have denoted the limit of decomposability with horizontal lines. We conclude that in four dimensions, $n = 5$ is precisely decomposable, but $n = 4$ is not. We now also see the deeper reason for this: in spite of there being 20 coefficients (one more than the minimum of 19), only 17 independent combinations can actually be formed. We also see that for sufficiently large n the number of independent combinations of coefficients saturates at $N(d, 1)$ as it ought to. We conclude that in d dimensions, an iGraph of order $d + 1$ is precisely decomposable with linear terms, but one of order d is of course not.

In the OPP method [15], the linear terms are precisely the spurious terms. We have to note that our general linear terms are not exactly those. The spurious terms have a specific property leading to fewer tensor structures, and we give an example in the next chapter. Rewriting our general linear terms in terms of propagators and spurious terms, we see that we decompose a pentagon into boxes *and* triangles (like in [4]and [15] for example). It can be checked that the triangles always cancel, and therefore the decomposition is actually unique.

At this point it must be pointed out that in all cases where a decomposition is possible in principle, we actually have obtained a solution for, the system (1.12). Once a would-be solution is found, it can easily be tested by evaluating eq. (1.3) for additional random values of the loop momentum[6] This 'global 1=1 test' then verifies this solution, where of course the equality

[6]In a certain sense, eq. (1.3) implies an *infinite* number of linear equations, of which we take ξ and solve them.

is supposed to hold only up to the precision used.

1.4 Reduction with coefficients of higher order

An iGraph of order $d+1$ becomes decomposable upon the inclusion of linear terms. As we explained before, a further decomposition is not possible. By including higher-rank tensors, such as quadratic or even cubic terms we can see that this is actually the case and we indeed do not get them. We give some tables with our findings in cases with coefficients of higher order.

	linear	quadratic	cubic	quartic
$d=2$				
$n=2$	6-0	21-1	20-3	30-6
3	9-1	18-5	30-11	45-19
4	12-4	18-5	40-21	60-34
5	15-7	30-17	50-31	
6	18-10	36-23	60-41	
$d=3$				
$n=2$	8-0	20-1	40-4	70-10
3	12-1	30-6	60-17	105-36
4	16-3	40-14	80-35	140-69
5	20-7	50-24	100-55	
6	24-11	60-34	120-75	
$d=4$				
$n=2$	10-0	30-1	70-5	
3	15-1	45-7	105-24	
4	20-3	60-17	140-52	280-121
5	25-6	75-30	175-85	350-189
6	30-11	90-45	210-120	
$d=5$				
$n=2$	12-0	42-1	112-6	
3	18-1	63-8	168-32	
4	24-3	84-20	224-72	
5	30-6	105-36	280-121	
6	36-10	126-55	336-175	

The rank of M from the inclusion of quadratic,
cubic, and quartic terms in the functions T.

We have now finished the discussion for scalar one-loop iGraphs, that have unity for their numerator. Let us regard and iGraph of order n with a nontrivial numerator, for instance

$$\frac{q \cdot k}{D_1 D_2 \cdots D_N} \quad .$$

Now, we can always arrange for ${p_1}^\mu = 0$ by a suitable shift of the loop momentum, and write the vector k^μ as

$$k^\mu = \omega^\mu + 2 \sum_{j=2}^{n} \zeta_j \, {p_j}^\mu \quad , \tag{1.13}$$

where the ζ's are fixed numbers and $\omega \cdot p_j = 0$ for all j (if $n \geq d+1$ then ω^μ simply vanishes). We can then write

$$(q \cdot k) = (q \cdot \omega) + \sum_{j=2}^{n} \zeta_j \, (D_j - D_1 - \mu_j + \mu_1) \tag{1.14}$$

so that this nonscalar iGraph decomposes into scalar iGraphs of order n and $n-1$, plus possibly a spurious term about which we do not worry since it integrates to zero. Our treatment of the scalar case is therefore sufficiently general.

Chapter 2

The OPP method

2.1 Introduction

So far,we have proved that any scalar integral of $d+1$ order in d dimensions can be decomposed to a set of integrals of at least one order less. That means that one can always find solutions for coefficients that satisfy eq. (1.3) if these coefficients are at least linear in the loop momentum. After solving for these coefficients one can always reduce further (i.e. à la Passarino-Veltman [3]) linear integrals of order d to scalar of the same and lower order.

In the OPP method (for a complete list see [15]-[23]) any one-loop amplitude (any integral of order n and numerator N(q)) is decomposed at once to scalar integrals of order d,$d-1$,...,1. Actually, what we proved before by looking at the minimum number of denominators that can be decomposed for a scalar integral is the starting point of an OPP method. Before we actually present the method let us refer for a moment to a paper by Pittau and del Aguila [27]. Having as a starting point a general tensor integral, the authors there construct a specific basis of four massless complex vectors $(l_1, ...l_4)$[1]. Two of them $(l_1, l_2\ )$ are written in terms of two independent external momenta and the other two (l_3, l_4) are taken to be orthogonal to them by using spinor techniques. Writting now the loop momentum in that basis, they manage to split it in components that reconstruct denominators (and thus lower the rank of the original tensor integral) and components in the l_3, l_4 plane that don't. However, a rank two tensor in this basis is proved to either reconstruct completely denominators, either be proportional to a rank one

[1] They work in $d=4$ or around it $(d=4+\epsilon)$

tensor times a denominator. It is obvious that with this method iteratively one can start from a general tensor integral and end up with a rank one tensor integral. This specific basis was later used in the OPP method [15] in order to find a list of all possible spurious terms at one-loop and because it can be implemented efficiently numerically for the solutions of the OPP method. However, for presenting the method itself it is not necessary to use it and in fact we are not going to. We will present the list of spurious terms in a different way than in [15].

2.2 The Master Formula

Once we discussed possible reductions in various dimensions we move to the OPP method. It is clear that such a method can be used in all dimensions. For the moment we stick to four dimensions unless otherwise explicitely mentioned. The integrand of any m-point one loop amplitude can be written as [27]

$$A(q) = \frac{N(q)}{D_0 D_1 \cdots D_{m-1}}\,, \quad D_i = (q+p_i)^2 - m_i^2\,, \tag{2.1}$$

Then, according to the OPP method, the numerator $N(q)$ can be decomposed as

$$\begin{aligned} N(q) = & \sum_{i_o<i_1<i_2<i_3}^{m-1} [d(i_o i_1 i_2 i_3) + \tilde{d}(q, i_o i_1 i_2 i_3)] \prod_{i\neq i_o,i_1,i_2,i_3}^{m-1} D_i + \\ & \sum_{i_o<i_1<i_2}^{m-1} [c(i_o i_1 i_2) + \tilde{c}(q, i_o i_1 i_2)] \prod_{i\neq i_o,i_1,i_2}^{m-1} D_i + \\ & \sum_{i_o<i_1}^{m-1} [b(i_o i_1) + \tilde{b}(q, i_o i_1)] \prod_{i\neq i_o,i_1}^{m-1} D_i + \\ & \sum_{i_o}^{m-1} [a(i_o) + \tilde{a}(q, i_o)] \prod_{i\neq i_o}^{m-1} D_i + \\ & \tilde{P}(q) \prod_{i}^{m-1} D_i \end{aligned} \tag{2.2}$$

In eq. (2.2) the coefficients d, c, b, a are the ones that multiply boxes, triangles, bubbles and tadpoles respectively, while the tilded ones, are the so-called spurious terms and as we see, they have q dependence. Although in the discussion before we talked about linear terms for the reduction of a pentagon to boxes, in principle, the spurious terms can be also of higher rank in the loop momentum. That is indeed the case of the triangle and the bubble spurious terms. Since spurious terms vanish upon integration what one really needs to calculate is the d, c, b, a coefficients only. However, this is not so easy and normally calculation of some of these coefficients needs at the same time calculation of the spurious part.

Another thing we assume is that we work in a renormalizable gauge. That means that for any diagram with m denominators, the numerator can be a tensor up to rank m. Notice that the numerator of eq. (2.1) comes from a collection of diagrams. That means that it can be up to rank $2m-1$ in a renormalizable gauge[2]. The $\tilde{P}(q)$ term is already of a higher order (at least $2m$) and since it's the only term of that order in a renormalizable gauge is not there. Working in this gauge also limits the number of the rest of the spurious terms.

2.3 List of one-loop spurious terms

We now give the list of one loop spurious terms from 4 up to 1-point functions in a renormalizable gauge. By the way we construct them, it is obvious that there are no spurious terms for higher cases (Pentagons, Hexagons, ...). The terms we describe here times a different coefficient for every box,triangle and so on, are the ones that enter in the OPP master formula.

2.3.1 Spurious terms for Boxes

We consider the integral for a general one loop box

$$\int d^4q D_4(p_1,p_2,p_3,p_4) = \int d^4q \frac{1}{D(q+p_1)D(q+p_2)D(q+p_3)D(q+p_4)} \quad (2.3)$$

We shift the loop momentum $q \to q - p_1$ and rename the other momenta $p_2 - p_1 = \hat{p}_1$, $p_3 - p_1 = \hat{p}_2$ and $p_4 - p_1 = \hat{p}_3$. The integral now becomes

[2]When a tadpole with a rank one numerator is multiplied and divided by the rest of $m-1$ denominators to be written at the form of eq. (2.1)

$$\int d^4q \frac{1}{D(q)D(q+\hat{p}_1)D(q+\hat{p}_2)D(q+\hat{p}_3)} \tag{2.4}$$

We remark here that the original integral depends on 4 momenta while we wrote it now as an integral depending on 3 momenta. The reason is that the momenta are not independent since we have the freedom of shifting the loop momentum. We define

$$\epsilon^{\mu\nu\rho\sigma} a_\mu b_\nu c_\rho d_\sigma = \epsilon(a,b,c,d) \tag{2.5}$$

for any vectors a,b,c,d, where $\epsilon^{\mu\nu\rho\sigma}$ is the Levi-Civita tensor.

The Levi-Civita tensor is fully antisymmetric in its Lorentz indices and is defined in 4 dimensions (in Minkowski space) by

$$\epsilon^{0123} = -\epsilon_{0123} = 1 \tag{2.6}$$

We list some of its most useful properties

$$\begin{aligned}
&\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\sigma} = -24 \\
&\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\rho\kappa} = -6\delta^\sigma_\kappa \\
&\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\nu\kappa\lambda} = -2\left(\delta^\rho_\kappa\delta^\sigma_\lambda - \delta^\sigma_\kappa\delta^\rho_\lambda\right) \\
&\epsilon^{\mu\nu\rho\sigma}\epsilon_{\mu\alpha\beta\gamma} = -(\delta^\nu_\alpha\delta^\rho_\beta\delta^\sigma_\gamma - \delta^\nu_\alpha\delta^\sigma_\beta\delta^\rho_\gamma + \delta^\rho_\alpha\delta^\sigma_\beta\delta^\nu_\gamma - \\
&\delta^\rho_\alpha\delta^\nu_\beta\delta^\sigma_\gamma + \delta^\sigma_\alpha\delta^\nu_\beta\delta^\rho_\gamma - \delta^\sigma_\alpha\delta^\rho_\beta\delta^\nu_\gamma) \\
&\epsilon(a,b,c,d)^2 = -G(a,b,c,d)
\end{aligned} \tag{2.7}$$

, where $G(a,b,c,d)$ the Gram determinant of the vectors a, b, c, d. As in [4] it is easy to see that

$$\int d^4q \frac{\epsilon(q,\hat{p}_1,\hat{p}_2,\hat{p}_3)}{D(q)D(q+\hat{p}_1)D(q+\hat{p}_2)D(q+\hat{p}_3)} = 0 \tag{2.8}$$

The proof is the same as in [4]. We repeat the proof since we are going to make use of it many times in the following cases. In the spirit of [3]

$$\int d^4q \frac{q_\mu}{D(q)D(q+\hat{p}_1)D(q+\hat{p}_2)D(q+\hat{p}_3)} = a\hat{p}_{1\mu} + b\hat{p}_{2\mu} + c\hat{p}_{3\mu} \qquad (2.9)$$

Contracting the above expression with $v^\mu = \epsilon^\mu(\hat{p}_1, \hat{p}_2, \hat{p}_3)$ we get 0 from the antisymmetric property of the Levi-Civita tensor. The vector v^μ is orthogonal to $\hat{p}_1, \hat{p}_2, \hat{p}_3$ vectors by construction and thus $q.v$ is a spurious term for the 4-point function above. We could investigate for spurious terms with higher powers of q. Working in a renormalizable gauge we could in principle have up to quartic spurious terms for a box.

We consider i.e. a quadratic polynomial in q. We investigate the integral

$$\int d^4q \frac{q_\mu v^\mu q_\nu v^\nu}{D(q)D(q+\hat{p}_1)D(q+\hat{p}_2)D(q+\hat{p}_3)} \qquad (2.10)$$

From Lorentz invariance this integral can only be proportional to combinations of $\hat{p}_{i\mu}\hat{p}_{j\nu}$ and $g_{\mu\nu}$. Any momenta contracting v will give zero since v is orthogonal to the other three momenta. The only interesting combination is when contracting $g_{\mu\nu}$ with $v^\mu v^\nu$ which doesn't give zero and thus is not a spurious term.

Moreover,from the completeness relation and by taking $B^{\mu\nu}$ to be some combination of $\hat{p}_1, \hat{p}_2$ and $\hat{p}_3$

$$g^{\mu\nu} = B^{\mu\nu}(\hat{p}_1, \hat{p}_2, \hat{p}_3) + v^\mu v^\nu \qquad (2.11)$$

which means that the combination $q_\mu v^\mu q_\nu v^\nu$ reconstructs denominators and no higher power of q contributes in the case of a box spurious term. The only spurious term for boxes is $q.v$ (something we expected before since with only linear terms we were able to decompose a pentagon into boxes). To get now the spurious term for the general box we defined in the begining we have to shift the loop momentum back to get the original propagators. Then

$$\begin{aligned}&\epsilon(q, \hat{p}_1, \hat{p}_2, \hat{p}_3) \to \epsilon(q+p_1, p_2-p_1, p_3-p_1, p_4-p_1) = \\ &\epsilon(q+p_1, q+p_2, q+p_3, q+p_4)\end{aligned} \qquad (2.12)$$

We then write

$$S(p_1, p_2, p_3, p_4) = \epsilon(q + p_1, q + p_2, q + p_3, q + p_4) \tag{2.13}$$

as the only spurious term for a general box integral.

2.3.2 Spurious terms for Triangles

We start again with a general triangle

$$\int d^4q D_3(p_1, p_2, p_3) = \int d^4q \frac{1}{D(q + p_1)D(q + p_2)D(q + p_3)} \tag{2.14}$$

and shift the loop momentum $q \to q - p_1$ and rename the other momenta $p_2 - p_1 = \hat{p}_1$,$p_3 - p_1 = \hat{p}_2$ so that the integral becomes

$$\int d^4q \frac{1}{D(q)D(q + \hat{p}_1)D(q + \hat{p}_2)} \tag{2.15}$$

We can find now 2 independent vectors orthogonal to $\hat{p}_1$ and $\hat{p}_2$

$$v_1^\mu = \epsilon^\mu(\hat{p}_1, \hat{p}_2, a) \tag{2.16}$$

and

$$v_2^\mu = \epsilon^\mu(\hat{p}_1, \hat{p}_2, v_1) \tag{2.17}$$

for any vector a independent of $\hat{p}_1, \hat{p}_2$, so that $q.v_1$ and $q.v_2$ are spurious.

Because of the existence of 2 independent vectors that give spurious terms in the case of triangles we can now construct spurious terms of higher power in q. A triangle can have up to cubic power in q spurious terms as described before.

Looking for quadratic terms, it is easy to prove that

$$\int d^4q \frac{v_1^\mu v_2^\nu q_\mu q_\nu}{D(q)D(q + \hat{p}_1)D(q + \hat{p}_2)} = 0 \tag{2.18}$$

By construction v_1 and v_2 are orthogonal and contracting them with the $g_{\mu\nu}$ term coming from the integral makes the integral above vanish. Terms coming from $v_1^\mu v_1^\nu$ and $v_2^\mu v_2^\nu$ do not vanish. However, we can find combinations and get one more independent quadratic spurious term

$$(v_2^2 v_1^\mu v_1^\nu - v_1^2 v_2^\mu v_2^\nu) q_\mu q_\nu \tag{2.19}$$

For the cubic terms we have much more options since all 8 combinations

$$v_1^\mu v_1^\nu v_1^\rho, v_1^\mu v_1^\nu v_2^\rho, ..., v_2^\mu v_2^\nu v_2^\rho \tag{2.20}$$

multiplied with $q_\mu q_\nu q_\rho$ give vanishing integrals. Since the number of q's is odd, there will always be a momentum arising from the integral that contracts one of the $v's$. They are not all independent though since the combination

$$(v_2^2 v_1^\mu v_1^\nu + v_1^2 v_2^\mu v_2^\nu) q_\mu q_\nu \tag{2.21}$$

reconstructs denominators. There are 2 cubic independent spurious terms in the case of triangles given by the following expressions

$$[3v_2^2 v_1^\mu v_1^\nu v_1^\rho - v_1^2(v_1^\mu v_2^\nu v_2^\rho + v_2^\mu v_1^\nu v_2^\rho + v_2^\mu v_2^\nu v_1^\rho)] q_\mu q_\nu q_\rho \tag{2.22}$$

and

$$[v_2^2(v_1^\mu v_1^\nu v_2^\rho + v_1^\mu v_2^\nu v_1^\rho + v_2^\mu v_1^\nu v_1^\rho) - 3v_1^2 v_2^\mu v_2^\nu v_2^\rho] q_\mu q_\nu q_\rho \tag{2.23}$$

These are all the spurious terms we have in the case of triangles.We now have to shift the loop momenta back to the original ones. We first define

$$A = \frac{v_2^2}{v_1^2} = [(p_2 - p_1)(p_3 - p_1)]^2 - (p_2 - p_1)^2 (p_3 - p_1)^2 \tag{2.24}$$

and we have the following spurious terms

$$\begin{aligned}
&S_1 = \epsilon(q+p_1, q+p_2, q+p_3, \alpha)\\
&S_2 = \epsilon^{\mu}(q+p_1, q+p_2, q+p_3)\epsilon_{\mu}(p_2-p_1, p_3-p_1, \alpha)\\
&S_3 = AS_1^2 - S_2^2\\
&S_4 = S_1 S_2\\
&S_5 = -3AS_1S_1S_1 + S_1S_2S_2 + S_2S_1S_2 + S_2S_2S_1\\
&S_6 = -3S_2S_2S_2 + A(S_2S_1S_1 + S_1S_2S_1 + S_1S_1S_2)
\end{aligned} \tag{2.25}$$

2.3.3 Spurious terms for Bubbles

We start with the bubble integral

$$\int d^4q D_2(p_1, p_2) = \int d^4q \frac{1}{D(q+p_1)D(q+p_2)} \tag{2.26}$$

and shift the loop momentum $q \to q - p_1$ and rename $p_2 - p_1 = \hat{p}_1$ so that the integral becomes

$$\int d^4q \frac{1}{D(q)D(q+\hat{p}_1)} \tag{2.27}$$

We can construct 3 independent vectors orthogonal to each other and to $\hat{p}_1$

$$v_1^{\mu} = \epsilon^{\mu}(a_1 a_2 \hat{p}_1) \tag{2.28}$$

$$v_2^{\mu} = \epsilon^{\mu}(a_3 \hat{p}_1 v_1) \tag{2.29}$$

$$v_3^{\mu} = \epsilon^{\mu}(\hat{p}_1 v_1 v_2) \tag{2.30}$$

such as $q.v_1$, $q.v_2$ and $q.v_3$ are spurious when integrated over $D(q)D(q+\hat{p}_1)$,for any vector a_1, a_2, a_3.

As before, searching for quadratic spurious terms we see that combinations such as

$$v_1^\mu v_2^\nu q_\mu q_\nu, \quad v_1^\mu v_3^\nu q_\mu q_\nu, \quad v_2^\mu v_3^\nu q_\mu q_\nu \tag{2.31}$$

are also spurious. Once again,$v_1^\mu v_1^\nu q_\mu q_\nu$, $v_2^\mu v_2^\nu q_\mu q_\nu$ and $v_3^\mu v_3^\nu q_\mu q_\nu$ do not integrate to zero but we can find two more independent quadratic in q spurious terms, namely

$$(v_2^2 v_1^\mu v_1^\nu - v_1^2 v_2^\mu v_2^\nu) q_\mu q_\nu \tag{2.32}$$

$$(v_3^2 v_1^\mu v_1^\nu - v_1^2 v_3^\mu v_3^\nu) q_\mu q_\nu \tag{2.33}$$

We have

$$B = \frac{v_2^2}{v_1^2} = -(p_2 - p_1)^2 a_3^2 + [(p_2 - p_1).a_3]^2 + \frac{(p_2 - p_1)^2 (v_1.a_3)^2}{v_1^2} \tag{2.34}$$

$$v_3^2 = -(p_2 - p_1)^2 B v_1^4 \tag{2.35}$$

By shifting the loop momentum back we get the spurious terms for the bubble integral

$$\begin{aligned}
&S_1 = \epsilon(q + p_1, q + p_2, a_1, a_2) \\
&S_2 = \epsilon^\mu(q + p_1, q + p_2, a_3)\epsilon_\mu(p_2 - p_1, a_1, a_2) \\
&S_3 \epsilon^{\mu\nu}(q + p_1, q + p_2)\epsilon_\mu(p_2 - p_1, a_1, a_2)\epsilon_{\nu\rho}(p_2 - p_1, a_3)\epsilon^\rho(p_2 - p_1, a_1, a_2) \\
&S_4 = S_1 S_2 \\
&S_5 = S_2 S_3 \\
&S_6 = S_1 S_3 \\
&S_7 = v_2^2 S_1^2 - v_1^2 S_2^2 \\
&S_8 = v_3^2 S_1^2 - v_1 S_3^2
\end{aligned} \tag{2.36}$$

2.3.4 Spurious terms for Tadpoles

We have the tadpole

$$\int d^4q \frac{1}{D(q+p_1)} \tag{2.37}$$

After shifting out p_1 a tadpole does not depend on external momenta. Therefore, any integral of the form

$$\int d^4q \frac{q^\mu}{D(q)} = 0 \tag{2.38}$$

We just have to find 4 vectors orthogonal to each other

$$v_1^\mu = \epsilon^\mu(a_1, a_2, a_3) \tag{2.39}$$

$$v_2^\mu = \epsilon^\mu(a_1, a_2, v_1) \tag{2.40}$$

$$v_3^\mu = \epsilon^\mu(a_1, v_1, v_2) \tag{2.41}$$

$$v_4^\mu = \epsilon^\mu(v_1, v_2, v_3) \tag{2.42}$$

for any vector a_1, a_2, a_3. When these 4 vectors contract with q_μ they produce 4 independent spurious terms for the tadpoles. Shifting back,we get

$$\begin{aligned} S_1 &= (q+p_1).v_1 \\ S_2 &= (q+p_1).v_2 \\ S_3 &= (q+p_1).v_3 \\ S_4 &= (q+p_1).v_4 \end{aligned} \tag{2.43}$$

The way we presented the spurious terms, using a basis similar to the van Neerven-Vermaseren one [4], differs from the way presented in [15]. The number of independent spurious terms we have in each case is the same, though, as we can see by comparing the two methods.

2.4 Solving for coefficients in the Master formula

In eq. (2.2) we wrote the numerator of any one-loop integrand in terms of denominators. By comparing polynomials in the loop momentum (left-hand side with right-hand side of eq. (2.2)) this is always possible. As the next step we must find solutions for these coefficients.

Since the reduction is valid for any value of the loop momentum we can always write eq. (2.2) for as many different values of q as the number of coefficients and solve the system. However, for an integrand with m denominators the number of coefficients without even taking into account the spurious terms is given by $\frac{m}{24}(m^3 - 2m^2 + 11m + 14)$. Then one has to invert huge determinants (i.e. for $m = 7$ a 98×98 determinant). It is obvious that taking into account the spurious terms (which are actually even more) one has to find a better strategy to address this problem.

This better strategy is related with the values of q that are used to solve the system. For some clever choices of q coefficients can be isolated and calculated easily, then be further used to calculate the rest. The system becomes triangular in that sense. These specific values of q are those that put denominators to zero. We'll explain in detail how by using such values the system is solved order by order.

2.4.1 Coefficients of the 4-point functions

Let us choose 4 denominators i.e. D_0, D_1, D_2, D_3 and find a q such that

$$D_0 = D_1 = D_2 = D_3 = 0 \tag{2.44}$$

Since we are in 4 dimensions, the loop momentum has 4 components and eq. (2.44) is uniquely solved. Actually, since a denominator is quadratic in the loop momentum,eq. (2.44) has exactly two solutions that we call q^+ and q^-. Now, we plug this values for q in eq. (2.2). There is only one term that does not contain any of the propagators above. Obviously all the other terms vanish and the master formula becomes:

$$N(q^\pm) = [d(0,1,2,3) + \tilde{d}(q^\pm, 0, 1, 2, 3)] \prod_{i \neq 0,1,2,3}^{m-1} D_i(q^\pm) \tag{2.45}$$

A comment is in order here. It is essential that boxes have only a linear spurious terms and eq. (2.44) has exactly two solutions due to its quadratic nature. Using these two solutions we can solve eq. (2.45) to obtain d and $\tilde{d}$. Once we know the q dependence of the spurious term we can write :

$$\tilde{d}(q^{\pm}, 0, 1, 2, 3) = \tilde{d}(0, 1, 2, 3)S(q^{\pm})$$

We also define

$$R(q^{\pm}) = \frac{N(q^{\pm})}{\prod_{i \neq 0,1,2,3}^{m-1} D_i(q^{\pm})}$$

As solutions we get

$$\begin{aligned} d(0,1,2,3) &= \frac{R(q^-)S(q^+) - R(q^+)S(q^-)}{S(q^+) - S(q^-)} \\ \tilde{d}(0,1,2,3) &= \frac{R(q^+) - R(q^-)}{S(q^+) - S(q^-)} \end{aligned} \tag{2.46}$$

Of course,by choosing all possible $\begin{pmatrix} m \\ 4 \end{pmatrix}$ set of 4 denominators to vanish we find different values of q and we can extract using them one by one all pairs of d and $\tilde{d}$ coefficients.

2.4.2 Coefficients of the 3-point functions

We have so far calculated all 4-point functions coefficients. Now we choose values of q such that 3 of the denominators vanish while the rest don't. We take i.e.

$$D_0 = D_1 = D_2 = 0 \ \ and \ \ D_i \neq 0, \ \ for \ i \neq 0, 1, 2 \tag{2.47}$$

In eq. (2.47) now the solutions we get are not unique as in the previous case. From 3 equations the 4 dimensional loop momentum cannot be completely determined. That allows us to find numerous q that satisfy eq. (2.47). That is also essential for the method since triangles have more than one spurious terms (they have 6) and 7 values of q are needed to determine the c and $\tilde{c}$ coefficients. Let assume that q_0 is one of the solutions of eq. (2.47) and we plug it in eq. (2.2).

$$N(q_0) = \sum_{2<i_3} [d(0,1,2,i_3) + \tilde{d}(q_0,0,1,2,i_3)] \prod_{i\neq 0,1,2,i_3}^{m-1} D_i(q_0) +$$
$$[c(0,1,2) + \tilde{c}(q_0,0,1,2)] \prod_{i\neq 0,1,2}^{m-1} D_i(q_0) \tag{2.48}$$

All the other terms vanish since they contain at least one of the propagators that vanish for q_0. We see now what is meant by solving the system order by order. Once we have solved for the d coefficients already,only the $c(0,1,2)$ and the $\tilde{c}(q_0,0,1,2)$ are still unknown in eq. (2.48). Notice that by $\tilde{c}$ we mean basically six different terms with different coefficient in front. Finding 7 such q_0 we are able from eq. (2.48) to determine them all. Then we repeat the proceedure for all *binom*3 different cases of an eq. (2.47) type of system.

In [15] and [19] one can find a more detailed analysis on the solutions of these systems, suitable for implementation in a computer code.

2.4.3 Coefficients of the 2-point functions

Having determined all d,$\tilde{d}$, c and $\tilde{c}$ coefficients we can now find values of q such that 2 propagators vanish and the rest do not. We choose i.e.

$$D_0 = D_1 = 0 \ \ and \ \ D_i \neq 0, \ \ for \ i \neq 0,1 \tag{2.49}$$

Since the equations are even less now, again there is an infinite number of q that satisfy eq. (2.49). Plugging such a q in eq. (2.2) we get :

$$N(q_0) = \sum_{1<i_2<i_3} [d(0,1,i_2,i_3) + \tilde{d}(q_0,0,1,i_2,i_3)] \prod_{i\neq 0,1,i_2,i_3}^{m-1} D_i(q_0) +$$
$$\sum_{1<i_2} [c(0,1,i_2) + \tilde{c}(q_0,0,1,i_2)] \prod_{i\neq 0,1,i_2}^{m-1} D_i(q_0) +$$

$$[b(0,1) + \tilde{b}(q_0,0,1)] \prod_{i \neq 0,1}^{m-1} D_i(q_0) \tag{2.50}$$

and all the rest vanish since they contain at least one of D_0 or D_1. Once again,by $\tilde{b}$ we mean all 8 2-point spurious terms with different coefficients in front. We find 9 values of that satisfy eq. (2.49) and then repeat the proceedure for all *binom*2 pairs of propagators to determine all b and $\tilde{b}$ coefficients.

2.4.4 Coefficients of the 1-point functions

The only thing that is left to be determined is the tadpole coefficients. The freedom of choosing a loop momentum that makes only one denominator vanish is even higher in this case and again we choose a q such as i.e

$$D_0 = 0 \;\; and \;\; D_i \neq 0, \;\; for \; i \neq 0 \tag{2.51}$$

We substitute this value of q in eq. (2.2) and we get

$$\begin{aligned}
N(q_0) = &\sum_{0<i_1<i_2<i_3} [d(0,i_1,i_2,i_3) + \tilde{d}(q_0,0,i_1,i_2,i_3)] \prod_{i \neq 0,i_1,i_2,i_3}^{m-1} D_i(q_0) + \\
&\sum_{0<i_1<i_2} [c(0,i_1,i_2) + \tilde{c}(q_0,0,i_1,i_2)] \prod_{i \neq 0,i_1,i_2}^{m-1} D_i(q_0) + \\
&\sum_{0<i_1} [b(0,i_1) + \tilde{b}(q_0,0,i_1)] \prod_{i \neq 0,i_1}^{m-1} D_i(q_0) + \\
&[a(0) + \tilde{a}(q_0,0)] \prod_{i \neq 0}^{m-1} D_i(q_0)
\end{aligned} \tag{2.52}$$

Notice that in principle we don't have to calculate the $\tilde{a}$ coefficients since they don't participate further (contrary to the other spurious terms that are needed for the coefficients of the next order). Finding such q's for all propagators one can determine all a coefficients and thus finish with the 4 dimensional part of the reduction.

2.5 Connection with Unitarity methods

We saw above how one can decompose a general one-loop integrand and find the coefficients using the OPP method. Once again,one can use any values of q to solve for the coefficients of eq. (2.2). The way the authors of [15] chose has the advantage that the system that has to be solved becomes more easy since with the specific choices of q there is less mixing of the coefficients in the equations of the system. In that sense the system is triangular.

There is also a more physical reason to choose this proceedure and it is connected with Unitarity methods. By searching for q's that make propagators vanish is basically the same with putting propagators on-shell. Take for example the d coefficients. With the OPP method one can calculate them by putting to zero the four denominators of the relevant box. In [10] the authors show by working in $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory that the coefficients of box one-loop integrals can be given from quadruple cuts. It was a strong hint that for general cases and by taking into account all possible cuts one can extract all the coefficients for any one-loop amplitude decomposition. It was most probably ideas like this that led to the OPP method.

2.6 Pentagons to Boxes revisited

In the first chapter we refered to the work of Melrose [1] and van Neerven-Vermaseren [4] that showed that a pentagon in 4 dimensions is decomposed into a sum of boxes. Later,we explained by counting that in order to do so we have to add linear terms to our coefficients. We also saw that in this case from 25 coefficients we start,only 19 are independent and they are exactly enough to form the 19 tensor structures that we get. In this chapter we presented the OPP method that performs this reduction with the help of spurious terms which are also linear in the loop momentum in the case of boxes. However, one can immediately notice that if a pentagon is written in terms of boxes in the OPP method the coefficients one starts with are only 10. Two questions are in order. Firstly, how is it possible with only 10 coefficients to construct the 19 tensor structures and secondly, are these methods all equivalent, do they give a unique answer ?

2.6.1 Pentagons to Boxes with the OPP method

Let us start with the first question. By defining

$$D_n(p_1, p_2, ..., p_n) = \frac{1}{D(q+p_1)D(q+p_2)...D(q+p_n)} \tag{2.53}$$

we try to decompose this pentagon

$$D_5(p_1, p_2, p_3, p_4, p_5)$$

to the 5 following boxes

$$D_4(p_2, p_3, p_4, p_5), D_4(p_3, p_4, p_5, p_1), D_4(p_4, p_5, p_1, p_2)$$

$$D_4(p_5, p_1, p_2, p_3)D_4(p_1, p_2, p_3, p_4)$$

Since all 5 external momenta that enter in the pentagon are not independent we can always arange them such that

$$\sum_{i=1}^{5} p_i = 0$$

We write the spurious terms in the following form

$$\begin{aligned}&S_1 = \epsilon(q+p_2, q+p_3, q+p_4, q+p_5) = q_\mu(\epsilon^\mu(p_3, p_4, p_5) - \epsilon^\mu(p_2, p_4, p_5) + \\&\epsilon^\mu(p_2, p_3, p_5) + \epsilon^\mu(p_2, p_3, p_4)) + \epsilon(p_2, p_3, p_4, p_5) = \\&q_\mu A_1^\mu + B_1\end{aligned} \tag{2.54}$$

From the eq. (2.2) we have

$$\begin{aligned}&1 = [d_1 + \tilde{d}_1 S_1]D(q+p_1) + [d_2 + \tilde{d}_2 S_2]D(q+p_2) + ... + \\&[d_5 + \tilde{d}_5 S_5]D(q+p_5)\end{aligned} \tag{2.55}$$

We compare the polynomials of the two hand-sides of eq. (2.55). The $q^2 q_\mu$ terms must vanish for any value of q which means

$$\sum_i (\tilde{d}_i A_i^\mu) = 0 \tag{2.56}$$

Notice that the A_i add up to zero

$$\sum_{i=1}^{5} A_i = 0$$

which means that all $\tilde{d}_i$ are equal (from now on we call them $\tilde{d}$). Then we take a look at the quadratic in q parts of the right-hand side of eq. (2.55). They cancel as well but one has to be careful since they come from two terms, the q^2 and the $q^\mu q^\nu$ term. We look at the second term

$$\tilde{d} q_\mu \sum_{i=1}^{5} (q \cdot p_i) A_i^\mu$$

Inserting the Schouten Identity

$$\begin{aligned}\epsilon(p_1, p_2, p_3, p_4) q^\mu = \epsilon^\mu(p_2, p_3, p_4)(q \cdot p_1) - \epsilon^\mu(p_1, p_3, p_4)(q \cdot p_2) + \\ \epsilon^\mu(p_1, p_2, p_4)(q \cdot p_3) - \epsilon^\mu(p_1, p_2, p_3)(q \cdot p_4)\end{aligned} \tag{2.57}$$

and using the fact that the external momenta add up to zero we get

$$\tilde{d} q_\mu \sum_{i=1}^{5} (q \cdot p_i) A_i^\mu = -5 \tilde{d} q^2 \epsilon(p_1, p_2, p_3, p_4) \tag{2.58}$$

That is exactly the property that the spurious term have that make the solution to the system possible. The $q^\mu q^\nu$ terms all vanish owing to the Schouten identity except for the trace part proportional to q^2 solving 9 out of 10 equations in one go. The total number of nontrivial equations is therefore 10 and not 19 in this case and the system has a solution. To complete the story, we are left with 5 d_i and one $\tilde{d}$ which are uniquely now determined from the remaining 6 equations (constant part,q^2 term and q_μ term).

2.6.2 Comparison of the various decompositions

We move now to the second question about the uniqueness of a possible decomposition. In [4] a reduction of pentagon to boxes is performed with the introduction of spurious terms. This is exactly the same terms we used for the OPP reduction so one can conclude that these two reductions are identical. In [15] as an application of the OPP method, the authors prove that this specific reduction is equivalent to the one found in [1] and [28]. In [28] the following relation is proven

$$\begin{vmatrix} I^N & -I^{N-1}(0) & \cdots & -I^{N-1}(N-1) \\ 1 & Y_{00} & \cdots & Y_{0(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & Y_{(N-1)0} & \cdots & Y_{(N-1)(N-1)} \end{vmatrix} = 0 \tag{2.59}$$

for $N \geq 5$, where the I^N is the N-point scalar function

$$\int \frac{1}{D_0, \cdots, D_{N-1}}$$

and $I^{N-1}(i)$ the $N-1$ point function with the i-th propagator of the N-point missing.The Y_{ij} functions are defined by

$$Y_{ij} = m_i^2 + m_j^2 - (p_i - p_j)^2, \;\; for\; i, j = 0, \cdots, N \tag{2.60}$$

By expanding eq. (2.59) for $N = 5$ one gets

$$I^5 = -\sum_{i=0}^{4} \frac{det_i(Y^{(5)})}{det(Y^{(5)})} I^4(i) \tag{2.61}$$

where $Y^{(5)}$ is the 5×5 Y matrix and $det_i(Y^{(5)})$ is the determinant of the Y matrix where the elements of the i-th column are replaced by 1. In [15] using the specific basis they work with and the fact that for this example $N(q) = 1$ they rewrite eq. (2.46) as

$$d_i = \frac{1}{2}(\frac{1}{D_i(q^+)} + \frac{1}{D_i(q^-)}) \tag{2.62}$$

and manage to prove that

$$d_i = -\frac{det_i(Y^{(5)})}{det(Y^{(5)})} \tag{2.63}$$

The proof is based on the fact that in 4 dimensions more than 4 vectors are linearly dependent and thus, their Gram determinant vanishes. Taking the Gram determinant of q,p_1,p_2,p_3,p_4,p_5 to vanish they end up with a second order equation for D_i

$$D_i^2 + bD_i + c = 0$$

with $b = -2det_i(Y^{(5)})$ and $c = detY^{(5)})$. The two roots of this equation $D_i(q^+)$ and $D_i(q^-)$ then have

$$\begin{aligned} D_i(q^+) + D_i(q^-) &= b \\ D_i(q^+)D_i(q^-) &= c \end{aligned} \tag{2.64}$$

and from eq. (2.62), eq. (2.63) is proven.

We saw that the OPP method,using a rather simple but crucial example, reproduces the reductions known with the conventional approaches ([1],[4], [28]). However, we still have to compare the reduction of the OPP method with spurious terms with the reduction using general linear in q terms we showed in the previous chapter. In the general linear case, we have some freedom while solving the system. From the original 25 coefficients only 19 are independent so one can choose in advance to give some values (for example put to zero) to some coefficients and solve for the rest [3].

Let's assume two different decompositions, one with spurious terms and one with general linear terms. Since they both come from the same pentagon we have:

$$\sum_j [x_j + \sum_{k=1}^{4} x_{j,k}(q \cdot t_k)]D_j = \sum_j [d_j + \tilde{d}_j S_j]D_j = 1 \tag{2.65}$$

[3]This is not always a safe choice. For instance, if we try to put to zero all 5 constant coefficients the system does not have a solution

We divide now with all denominators and we integrate.The spurious terms vanish

$$\sum_j \int \frac{x_j + \sum_{k=1}^4 x_{j,k}(q \cdot t_k)}{D_4(j)} = \sum_j \int \frac{d_j}{D_4(j)}$$

where with $D_4(j)$ we mean the product of the four out of five denominators with the j-th absent. This is equivalent to

$$\sum_j \int \frac{x_j - d_j}{D_4(j)} = -\sum_j \int \sum_{k=1}^{4} x_{j,k} \frac{(q \cdot t_k)}{D_4(j)} \tag{2.66}$$

We can use Passarino-Veltman [3] in the right-hand side of eq. (2.66)

$$\int \frac{q^\mu}{D_4(j)} \propto a \int \frac{1}{D_4(j)} + \sum_l \int b_l \frac{1}{D_3(jl)}$$

where $D_3(jl)$ is the product of 3 propagators with the l propagator of the $D_4(j)$ missing.

Notice that after performing the Passarino-Veltman reduction also boxes appear in the right hand side of the equation that we move to the left side. It would seem that eq. (2.66) can now be written as

linear combination of boxes = linear combination of triangles

We compared numerically the coefficients we get from the two reductions that multiply the triangles. We noticed that all these coefficients vanish. Such a relation between boxes and triangles does not exist. We conclude that reducing a pentagon to boxes gives a unique answer independently if one uses spurious or general linear terms.

In the appendix A we present the analogous result in 2 dimensions in more detail. We show explicitely how with a choice of general linear terms one can get the OPP result. To make the proof easier we chose a specific basis for the linear terms where spurious terms inside can be identified immediately.

We presented the OPP method in great detail although there are still some things left to discuss. So far we worked strictly in 4 dimensions (with the exception of the cases we worked in other integer dimensions). In case one departs from 4 dimensions for the sake of dimensional regularization (i.e. work in $4+\epsilon$ dimensions) extra pieces arise that one has to calculate also. They are called Rational Terms and we will deal with them, also in great detail, in the next chapters.

Chapter 3

Rational Terms of one-loop Amplitudes

3.1 Introduction

Loop integrals often suffer from divergencies of ultraviolet (UV) or infrared (IR) nature. For ultraviolet divergencies, a naive counting of powers of loop momenta in the integral is most of the times enough to say if the integrals suffers from them or not. The infrared divergencies are mostly related to the presence of massless particles that lead to soft/collinear singularities. There are many ways to define such integrals when these singularities are present. The method that is mostly used is dimensional regularisation [29],[30]. The idea is that moving from four dimensions to $4+\epsilon$ dimensions, we introduce a parameter ϵ that regulates both types of divergencies and make the integral finite. At the end of the calculation the limit $\epsilon \to 0$ is taken.

After reducing large integrals with the help of the OPP reduction formula we still have to evaluate the remaining integrals. Therefore, we have to extend the method to an arbitrary dimension [1]. More precisely, when expanding the numerator of our amplitude in terms of denominators,if the loop momentum also lives in $4+\epsilon$ dimensions then extra pieces arise. These pieces are called Rational Terms and in the spirit of the OPP method we can classify them into two categories (R_1 and R_2).

[1]We already saw reduction in other integer dimensions. What we mean here is investigate what happens in case we are working in a dimension in a 'neighborhood' of four dimensions,i.e. $4+\epsilon$ dimensions

3.2 OPP method in $4+\epsilon$ dimensions

Departing from 4 dimensions eq. (2.1) becomes

$$\bar{A}(\bar{q}) = \frac{\bar{N}(\bar{q})}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}}\,, \quad \bar{D}i = (\bar{q}+p_i)^2 - m_i^2\,, \tag{3.1}$$

In the scheme we are going to follow, the loop momentum lives in $4+\epsilon$ dimensions and is denoted with a bar notation, while objects like external momenta and masses still live in 4 dimensions [2]. For purely ϵ-dimensional parts we use the tilde notation. We can decompose thus every object to 4 and ϵ-dimensional pieces:

$$\begin{aligned} \bar{q} &= q + \tilde{q} \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}} \\ \bar{g}_{\bar{\mu}\bar{\nu}} &= g_{\mu\nu} + \tilde{g}_{\tilde{\mu}\tilde{\nu}} \end{aligned} \tag{3.2}$$

When an n-dimensional index is contracted with a 4-dimensional vector, the 4-dimensional part is automatically selected

$$\bar{q}^2 = q^2 + \tilde{q}^2, \quad \bar{q}\cdot v = q\cdot v \tag{3.3}$$

We split now the numerator function in a 4-dimensional plus and ϵ dimensional part

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, q, \epsilon)\,. \tag{3.4}$$

and substitute the two parts in eq. (3.1). We have

$$\frac{N(q)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} + \frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}}$$

In the first term we see that there is mismatch in the dimensions between the numerator and the denominator. This mismatch induces rational terms

[2] There are different schemes where all objects become $4+\epsilon$ dimensional. In all the schemes we will mention the external momenta and the masses will be strictly 4-dimensional

of the type R_1 (together with the purely 4 dimensional part called Cut-Constructible (CC) as we shall see) while the second term contains the ϵ dimensional part of the numerator and will give rise to another part of the rational terms we call R_2.

Performing a complete one-loop calculation we therefore have

$$A = C.C. + R_1 + R_2 \tag{3.5}$$

3.3 The R_1 part

We start from the term

$$\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \tag{3.6}$$

Since we can rewrite each denominator as a 4 dimensional denominator plus a $\tilde{q}^2$

$$\bar{D}_i = D_i + \tilde{q}^2$$

we have

$$\frac{1}{\bar{D}_i} = \frac{\bar{Z}_i}{D_i}, \quad \bar{Z}_i = 1 - \frac{\tilde{q}^2}{\bar{D}_i} \tag{3.7}$$

We then substitute in eq. (3.6) to get

$$\frac{N(q)}{D_0 D_1 \cdots D_{m-1}} \bar{Z}_0 \bar{Z}_1 \cdots \bar{Z}_{m-1} \tag{3.8}$$

By writing eq. (3.6) we manage to show explicitely the pure 4-dimensional piece (coming from the 1 part of each $\bar{Z}_i$ and a part with a $\tilde{q}^2$ dependence. The first part is exactly the 4-dimensional part we have already discussed in the previous chapter and it is called Cut-Constructible part (CC). We already know how to calculate it by putting propagators to zero, by performing cuts, and that's where it takes the name from. The second part gives rise to what we call R_1 [18] after integrating the loop momentum out. More precisely

$$\mathrm{R}_1 \equiv \frac{1}{(2\pi)^4} \int d^n\, \bar{q} \frac{f(\tilde{q}^2, q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \tag{3.9}$$

In [17],[16] and [18] methods have been proposed how to calculate the R_1 part.

If we substitute in eq. (3.8) the numerator given in eq. (2.2) we have

$$\begin{aligned} \bar{A}(\bar{q}) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2} \bar{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{Z}_i \\ &+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i_0} \bar{D}_{i_1} \bar{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{Z}_i \\ &+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i_0} \bar{D}_{i_1}} \prod_{i \neq i_0, i_1}^{m-1} \bar{Z}_i \\ &+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i_0}} \prod_{i \neq i_0}^{m-1} \bar{Z}_i \end{aligned} \tag{3.10}$$

Then we get the R_1 part integrating all $\tilde{q}^2$ terms. However, this method has two drawbacks. The spurious terms are now multiplied by Z_i and it's not guaranteed that they remain spurious [3]. Secondly,when diagrams are lumped together in a single numerator by constructing common denominators, additional $\tilde{q}^2$ terms appear that lead to new rational parts. That threatens one of the most important properties of the OPP method which is reducing at once the whole amplitude and not just working at a diagram level. However, there is a better way to calculate the R_1 part that maintains the advantages and the properties of the OPP method.

The second method is to write eq. (2.2) directly in terms of $4+\epsilon$ dimensional objects and perform the expansion of the numerator using $4+\epsilon$ cuts. In other words, introduce in all OPP coefficients a $\tilde{q}^2$ dependence. This dependence enters since considering denominators in $4+\epsilon$ dimensions we basically introduce a mass shift

[3]The reason is that since these terms are multiplied with Z_i's they depend on more external momenta. The results of the integration are not orthogonal any more to the vectors we constructed with the help of the Levi-Civita tensors and thus they don't vanish

$$m_i^2 \to m_i^2 - \tilde{q}^2 \tag{3.11}$$

Notice that from Lorentz invariance this dependence does not change the spurious terms. Of course,extra integrals with powers of $\tilde{q}^2$ in the numerator are generated. We give a list of these integrals in the appendix B (together with more integrals that are relevant for the R_2 piece we will introduce in a while).

In [18] the $\tilde{q}^2$ dependece of the OPP coefficients is described. We use their example to explain how to find the coefficients when performing $4+\epsilon$-dimensional cuts. Of course we refer to the paper for more details.

They start by proving

$$\begin{aligned} b(ij;\tilde{q}^2) &= b(ij) + \tilde{q}^2 b^{(2)}(ij)\,, \\ c(ijk;\tilde{q}^2) &= c(ijk) + \tilde{q}^2 c^{(2)}(ijk)\,. \end{aligned} \tag{3.12}$$

Furthermore, by using eq. (3.11), the first line of eq. (2.2) in $4+\epsilon$ dimensions becomes

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) \equiv \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0i_1i_2i_3;\tilde{q}^2) + \tilde{d}(q;i_0i_1i_2i_3;\tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \,, \tag{3.13}$$

and the following expansion holds

$$\mathcal{D}^{(m)}(q,\tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q)\,, \tag{3.14}$$

where the last coefficient is independent on q

$$d^{(2m-4)}(q) = d^{(2m-4)}\,. \tag{3.15}$$

In practice, once the 4-dimensional coefficients have been determined, one simply redoes the fits for different values of $\tilde{q}^2$, in order to determine the $\tilde{q}^2$-parts of the coefficients. Using the results of the relevant integrals from the appendix we have

$$
\begin{aligned}
\mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\
&- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) . \qquad (3.16)
\end{aligned}
$$

3.4 The R_2 part

When going to $4+\epsilon$ dimensions we discussed how we can split the amplitude in a pure 4-dimensional piece and a rational part we call R_1. In order to finish completely with the one-loop amplitude there is a last piece left coming from the ϵ-dimensional part of the numerator

$$
\frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}
$$

We call this part R_2. We define

$$
R_2 = \frac{1}{(2\pi)^4} \int d^d \bar{q} \frac{\tilde{N}(\tilde{q}^2, q, \epsilon)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} = \frac{1}{(2\pi)^4} \int d^d \bar{q} \mathcal{R}_2 \qquad (3.17)
$$

Given a numerator of an amplitude (or a diagram) one can recognise the R_2 part by performing all algebraic manipulations in d dimensions and isolate the explicit ϵ or $\tilde{q}^2$ terms of the numerator. The first come from $\bar{q}^2 = q^2 + \tilde{q}^2$ terms while the latter can be produced only by $\bar{g}_{\bar{\mu}\bar{\nu}} \bar{g}^{\bar{\mu}\bar{\nu}}$ or $\bar{\gamma}^{\bar{\mu}} \Gamma \bar{\gamma}_{\bar{\mu}}$ contractions, where Γ any string of gamma matrices.

As we have seen before the R_1 part and the Cut-Constructible part of the amplitude are directly connected. However, the R_2 part needs a different treatment since it is not possible to put it at the same footing with them and reconstruct it completely numerically as was explained in [22]. Working in the basis introduced in [27] one can show that not enough information is present in the 4-dimensional part to reconstruct R_2.

For example, one could naively think that, by looking at any q^2 in the CC part, the $\tilde{q}^2$ dependence could be inferred via the replacement

$$
q^2 \to q^2 + \tilde{q}^2 . \qquad (3.18)
$$

However, such a dependence is impossible to reconstruct numerically, when remaining in 4 dimensions, as it can be illustrated by considering the following simple 3-point sub-amplitude:

$$\mathrm{A} \equiv \frac{1}{(2\pi)^4} \int d^n\, \bar{q} \frac{(q \cdot \ell_3)(q \cdot \ell_4)}{\bar{D}_0 \bar{D}_1 \bar{D}_2}\,, \tag{3.19}$$

where

$$\ell_3^\mu =< \ell_1|\gamma^\mu|\ell_2]\,, \quad \ell_4^\mu =< \ell_2|\gamma^\mu|\ell_1] \quad \text{with} \quad \ell_{1,2}^2 = 0\,. \tag{3.20}$$

From the one hand, the 4-dimensional numerator $(q \cdot \ell_3)(q \cdot \ell_4)$ in eq. 3.19 does not contain any q^2 to be continued through the replacement of eq. 3.18. On the other hand, it can be manipulated as follows

$$(q \cdot \ell_3)(q \cdot \ell_4) = 4\,(q \cdot \ell_1)(q \cdot \ell_2) - 2\,q^2\,(\ell_1 \cdot \ell_2)\,, \tag{3.21}$$

and now the shift of eq. 3.18 would produce a $\tilde{q}^2$ contribution, in disagreement with our previous finding.For the specific part one is forced to work analytically in d dimensions to obtain the R_2 contribution.

A practical way to determine R_2 is computing analytically, by means of Feynman diagrams, once and for all tree-level like Feynman rules, namely effective vertices like counterterms, by calculating the R_2 part coming from all possible one-particle irreducible Green functions of the theory at hand, up to four external legs. The fact that four external legs are enough to account for R_2 is guaranteed by the ultraviolet nature of the rational terms, proved in [31]. This property does not hold, instead, for R_1, that, diagram by diagram, can give non vanishing contributions to any one-particle irreducible m-point function, because, even when finite, the tensor integrals generating R_1 are eventually expressed, via tensor reduction, in terms of linear combinations of 1-loop scalar functions that can be ultraviolet divergent. This fact prevents the possibility of calculating a finite set of effective vertices reproducing R_1.

In case one performs this calculation in different regularization schemes, results may differ. In eq. 3.17 we assume the 't Hooft-Veltman (HV) scheme, while in the Four Dimensional Helicity scheme (FDH), any explicit ϵ dependence in the numerator function is discarded before integration, such that

$$\mathrm{R}_2\Big|_{FDH} = \frac{1}{(2\pi)^4} \int d^n\, \bar{q} \frac{\tilde{N}(\tilde{q}^2, q, \epsilon = 0)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}\,. \tag{3.22}$$

Before we proceed to the results of the effective R_2 vertices a comment is in order. The way we split the rational part in two pieces that we treat seperately is not the only way of calculating it. For example in [41] show that the numerator depends linearly on the number of dimensions at one loop, that means, by calculating it in two different (integer) dimensions they can calculate numerically the d dimensional part and obtain the rational terms. However,that would require the use of multidimensional spinors, vectors et.c. plus a double calculation of a one-loop amplitude. In our case, we mainly have to find these effective vertices once and for all. Then,the R_1 part is calculated together with the C.C. part while the R_2 part only needs an extra tree level calculation that adds zero in calculation time in comparison with the one-loop calculation.

Chapter 4

R_2 vertices for the Standard Model

The effective vertices for the R_2 part have been calculated and listed in [18], [21] and [22]. In these papers one can find effective vertices for QED,QCD and the Electroweak standard model repsectively providing the last tools needed to compute any one-loop standard model amplitude. We will present and discuss in detail mainly the Electroweak standard model case. For the QCD case we refer to one of the papers above [21].

4.1 Calculation

The 1-loop irreducible diagrams up to four external legs of the whole standard model is of the order 10.000. Therefore we considered first generic diagrams (diagrams with not specified particles inside,just type of particle) and we calculated the R_2 contributions of them for generic Feynman rules as well. The advantage is that in most cases it is easy to see if the generic diagrams will have a contribution or not. For example, a simple power counting in loop momenta in the numerator most of the times indicates if the generic diagram will lead or not to one of the integrals listed in appendix B. Once we calculated the generic diagrams we substituted specific Feynman rules (we used the ones in [28]) and got the effective vertices. In [32] we list some of the generic diagrams with an R_2 contribution.

For the calcultion itself we used mainly FORM [33] [1]. We performed the

[1] The results can be also found as a `FORM` output in

calculation in the 't Hooft-Feynman gauge [2].

A parameter λ_{HV} is introduced in our formulae such that $\lambda_{HV} = 1$ corresponds to the 't Hooft-Veltman scheme and $\lambda_{HV} = 0$ to the FDH scheme of eq. 3.22. For the different particles we used the following notation :

	symbol 1	symbol 2
e	l_1	e_1
μ	l_2	e_2
τ	l_3	e_3
ν_e	l_4	ν_1
ν_μ	l_5	ν_2
ν_τ	l_6	ν_3

Notation for leptons

	symbol 1	symbol 2
u	q_1	u_1
d	q_2	d_1
s	q_3	d_2
c	q_4	u_2
b	q_5	d_3
t	q_6	u_3

Notation for quarks

When appearing as external particles, l, ν_l, u and d stand for the three charged leptons, the three (massless) neutrinos, the three up-type quarks and the three down-type quarks, respectively. Effective vertices with external quarks are always understood to be diagonal in the color space. Finally, N_{col} is the number of colors and $V_{u_i d_j}$ are CKM matrix elements. Occasionally, combinations such as

$$\sum_{i,j=1}^{3} \left(V_{u_i d_j} V^{\dagger}_{d_j u_i} \right) = 3 \quad \text{and} \quad \sum_{i=1}^{3} 1 = 3$$

appear in our formulae. In such cases, we do not explicitly work out the sum in order to make our results also readable family by family.

http://www.ugr.es/local/pittau/CutTools.

[2] In the next chapter we will discuss gauge invariant issues for the rational part and there we also performed this calculation in different gauges as well

A last comment is in order with respect to our treatment of γ_5 in vertices containing fermionic lines. When computing all contributing Feynman diagrams, we pick up a “special” vertex in the loop and anticommute all γ_5's to reach it before performing the n-dimensional algebra, and, when a trace is present, we start reading it from this vertex. This treatment produces, in general, a term proportional to the totally antisymmetric ϵ tensor, whose coefficient may be different depending on the choice of the “special” vertex. However, when summing over all quantum numbers of each fermionic family, we checked that all contributions proportional to ϵ cancel.

4.2 Results

We present now the complete list of results for the Electroweak case [22].

4.2.1 Scalar-Scalar effective vertices

The generic effective vertex is

$$\boxed{S_1 \text{ - - - - - } \bullet \text{ - - - - - } S_2 \;=\; \frac{ie^2}{16\pi^2 s_w^2} C}$$

with the actual values of S_1, S_2 and C

$$H\chi \quad : \quad C \;=\; 0$$

$$HH \quad : \quad C \;=\; \frac{m_\phi^2}{4} + \frac{m_\chi^2}{8c_w^2} + \frac{1-12\lambda_{HV}}{4}\left(1+\frac{1}{2c_w^4}\right)m_W^2 - \left(1+\frac{1}{2c_w^2}\right)\frac{p^2}{12} + K$$

$$\chi\chi \quad : \quad C \;=\; \frac{m_\phi^2}{4} + \frac{m_H^2}{8c_w^2} + \frac{1-4\lambda_{HV}}{4}\left(1+\frac{1}{2c_w^4}\right)m_W^2 - \left(1+\frac{1}{2c_w^2}\right)\frac{p^2}{12} + K$$

$$\begin{aligned}\phi^-\phi^+ \quad : \quad C \;=\;& \frac{m_H^2+m_\chi^2}{8} + \frac{(3-4\lambda_{HV})\,c_w^4 - 2c_w^2 + \left(\frac{1}{2}-2\lambda_{HV}\right)}{c_w^4}\frac{m_W^2}{4} + \frac{m_\phi^2}{8c_w^2} \\ &- \left(1+\frac{1}{2c_w^2}\right)\frac{p^2}{12} + \frac{1}{2m_W^2}\left[\sum_{i=1}^{3}\left(m_{e_i}^2\left(m_{e_i}^2 - \frac{p^2}{3}\right)\right)\right.\end{aligned}$$

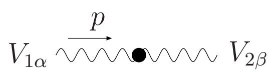
$$
+ \quad N_{\rm col} \sum_{i,j=1}^{3} \left(V_{u_i d_j} V^{\dagger}_{d_j u_i} \left(m^2_{u_i} + m^2_{d_j} \right) \left(m^2_{u_i} + m^2_{d_j} - \frac{p^2}{3} \right) \right) \Bigg] \quad (4.1)
$$

where

$$
K \;=\; \frac{1}{m_W^2} \left[\sum_{i=1}^{6} \left(m^2_{l_i} \left(m^2_{l_i} - \frac{p^2}{6} \right) \right) + N_{\rm col} \sum_{i=1}^{6} \left(m^2_{q_i} \left(m^2_{q_i} - \frac{p^2}{6} \right) \right) \right] \quad (4.2)
$$

4.2.2 Vector-Vector effective vertices

The generic effective vertex is



$$
= \frac{ie^2}{\pi^2} \left(C_1 p_\alpha p_\beta + C_2 g_{\alpha\beta} \right)
$$

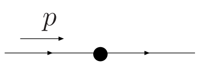
with the actual values of V_1, V_2, C_1 and C_2

$$
\begin{aligned}
AA \;:\; C_1 &= -\frac{1}{24} \lambda_{HV} \\
C_2 &= \frac{1}{8} \left[p^2 \left(\frac{1}{6} + \frac{\lambda_{HV}}{3} \right) - m_W^2 \right] - \frac{1}{4} \left[\sum_{i=1}^{6} \left(Q^2_{l_i} \left(m^2_{l_i} - \frac{1}{6} p^2 \right) \right) \right. \\
&+ \left. N_{\rm col} \sum_{i=1}^{6} \left(Q^2_{q_i} \left(m^2_{q_i} - \frac{1}{6} p^2 \right) \right) \right] \\
AZ \;:\; C_1 &= \frac{1}{24} \frac{c_w}{s_w} \lambda_{HV} \\
C_2 &= -\frac{1}{8} \frac{c_w}{s_w} \left[p^2 \left(\frac{1}{6} + \frac{\lambda_{HV}}{3} \right) - m_W^2 \right] + \frac{1}{4 c_w} \left[\sum_{i=1}^{6} \left(\left(\frac{Q_{l_i} I_{3 l_i}}{2 s_w} - Q^2_{l_i} s_w \right) \right. \right. \\
&\times \left. \left. \left(m^2_{l_i} - \frac{1}{6} p^2 \right) \right) + N_{\rm col} \sum_{i=1}^{6} \left(\left(\frac{Q_{q_i} I_{3 q_i}}{2 s_w} - Q^2_{q_i} s_w \right) \left(m^2_{q_i} - \frac{1}{6} p^2 \right) \right) \right] \\
ZZ \;:\; C_1 &= -\frac{1}{24} \frac{c_w^2}{s_w^2} \lambda_{HV} \\
C_2 &= \frac{1}{8} \frac{c_w^2}{s_w^2} \left[p^2 \left(\frac{1}{6} + \frac{\lambda_{HV}}{3} \right) - m_W^2 \right] + \frac{1}{4 c_w^2} \left[\sum_{i=1}^{6} \left(\left(Q_{l_i} I_{3 l_i} - \frac{I^2_{3 l_i}}{2 s_w^2} - Q^2_{l_i} s_w^2 \right) \right. \right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \quad \left(m_{l_i}^2 - \frac{1}{6}p^2\right)\Big) + N_{\rm col}\sum_{i=1}^{6}\left(\left(Q_{q_i}I_{3q_i} - \frac{I_{3q_i}^2}{2s_w^2} - Q_{q_i}^2 s_w^2\right)\left(m_{q_i}^2 - \frac{1}{6}p^2\right)\right)\Bigg]\\
W^-W^+ \quad &: \quad C_1 \quad = \quad -\frac{1}{24s_w^2}\lambda_{HV}\\
&\quad\;\; C_2 \quad = \quad \frac{1}{8s_w^2}\left[p^2\left(\frac{1}{6}+\frac{\lambda_{HV}}{3}\right) - m_W^2\right] - \frac{1}{32s_w^2}\left[\sum_{i=1}^{3}\left(m_{e_i}^2 - \frac{p^2}{3}\right)\right.\\
&\qquad\qquad + \left. N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_id_j}V_{d_ju_i}^\dagger\left(m_{u_i}^2 + m_{d_j}^2 - \frac{p^2}{3}\right)\right)\right] \qquad (4.3)
\end{aligned}
$$

4.2.3 Fermion-Fermion effective vertices

The generic effective vertex is

F_1  $\bar{F}_2 \;=\; \dfrac{ie^2}{\pi^2}\left[\left(C_-\Omega^- + C_+\Omega^+\right)\not{p} + C_0\right]\lambda_{HV}$

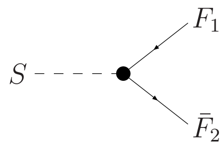
with the actual values of F_1, $\bar{F}_2$, C_-, C_+ and C_0

$$
\begin{aligned}
u\bar{u} \quad &: \quad C_- = \frac{1}{16}\frac{Q_u^2}{c_w^2}\\
&\quad\;\; C_+ = \frac{1}{16}\left(\frac{I_{3u}^2}{s_w^2c_w^2} - \frac{2Q_uI_{3u}}{c_w^2} + \frac{Q_u^2}{c_w^2} + \frac{1}{2s_w^2}\sum_{j=1}^{3}\left(V_{ud_j}V_{d_ju}^\dagger\right)\right)\\
&\quad\;\; C_0 = \frac{m_uQ_u}{8c_w^2}\left(Q_u - I_{3u}\right)\\
d\bar{d} \quad &: \quad C_- = \frac{1}{16}\frac{Q_d^2}{c_w^2}\\
&\quad\;\; C_+ = \frac{1}{16}\left(\frac{I_{3d}^2}{s_w^2c_w^2} - \frac{2Q_dI_{3d}}{c_w^2} + \frac{Q_d^2}{c_w^2} + \frac{1}{2s_w^2}\sum_{i=1}^{3}\left(V_{u_id}V_{du_i}^\dagger\right)\right)\\
&\quad\;\; C_0 = \frac{m_dQ_d}{8c_w^2}\left(Q_d - I_{3d}\right)\\
l\bar{l} \quad &: \quad C_- = \frac{1}{16}\frac{Q_l^2}{c_w^2}
\end{aligned}
$$

$$
\begin{aligned}
C_+ &= \frac{1}{16}\left(\frac{I_{3l}^2}{s_w^2 c_w^2} - \frac{2Q_l I_{3l}}{c_w^2} + \frac{Q_l^2}{c_w^2} + \frac{1}{2s_w^2}\right) \\
C_0 &= \frac{m_l Q_l}{8c_w^2}\left(Q_l - I_{3l}\right) \\
\nu_l\bar{\nu}_l \quad : \quad C_- &= 0 \\
C_+ &= \frac{1}{32s_w^2}\left(\frac{1}{2c_w^2} + 1\right) \\
C_0 &= 0
\end{aligned} \tag{4.4}
$$

4.2.4 Scalar-Fermion-Fermion effective vertices

The generic effective vertex is



$$= \frac{e^3}{\pi^2}\left(C_-\Omega^- + C_+\Omega^+\right)$$

with the actual values of S, F_1, $\bar{F}_2$, C_- and C_+

$$
\begin{aligned}
Hu\bar{u} \quad : \quad C_- &= \frac{im_u}{8m_W s_w}\left[\frac{(1+\lambda_{HV})\,Q_u^2}{2c_w^2} + \frac{1}{16s_w^2}\sum_{j=1}^{3}\left(V_{ud_j}V_{d_j u}^\dagger\right) + \frac{I_{3u}}{c_w^2}\left(\frac{I_{3u}}{8s_w^2}\right.\right. \\
&\quad - \left.\left.\frac{(1+\lambda_{HV})\,Q_u}{2}\right) + \frac{1}{16m_W^2 s_w^2}\sum_{j=1}^{3}\left(m_{d_j}^2 V_{ud_j}V_{d_j u}^\dagger\right)\right] \\
C_+ &= C_-
\end{aligned}
$$

$$
\begin{aligned}
Hd\bar{d} \quad : \quad C_- &= \frac{im_d}{8m_W s_w}\left[\frac{(1+\lambda_{HV})\,Q_d^2}{2c_w^2} + \frac{1}{16s_w^2}\sum_{i=1}^{3}\left(V_{u_i d}V_{du_i}^\dagger\right) + \frac{I_{3d}}{c_w^2}\left(\frac{I_{3d}}{8s_w^2}\right.\right. \\
&\quad - \left.\left.\frac{(1+\lambda_{HV})\,Q_d}{2}\right) + \frac{1}{16m_W^2 s_w^2}\sum_{i=1}^{3}\left(m_{u_i}^2 V_{u_i d}V_{du_i}^\dagger\right)\right] \\
C_+ &= C_-
\end{aligned}
$$

$$
\begin{aligned}
H l\bar{l} \quad : \quad C_- &= \frac{i m_l}{8 m_W s_w}\left[\frac{(1+\lambda_{HV})\,Q_l^2}{2c_w^2}+\frac{1}{16 s_w^2}+\frac{I_{3l}}{c_w^2}\left(\frac{I_{3l}}{8s_w^2}-\frac{(1+\lambda_{HV})\,Q_l}{2}\right)\right]\\
C_+ &= C_-
\end{aligned}
$$

$$
\begin{aligned}
H\nu_l\bar{\nu}_l \quad : \quad C_- &= 0\\
C_+ &= 0
\end{aligned}
$$

$$
\begin{aligned}
\chi u\bar{u} \quad : \quad C_- &= -\frac{m_u}{4 m_W s_w}\left[\frac{(1+\lambda_{HV})\,Q_u^2 I_{3u}}{2c_w^2}+\frac{1}{32 s_w^2}\sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_j u}\right)+\frac{I_{3u}}{c_w^2}\left(\frac{1}{32s_w^2}\right.\right.\\
&- \left.\left.\frac{(1+\lambda_{HV})\,Q_u I_{3u}}{2}\right)-\frac{1}{16 m_W^2 s_w^2}\sum_{j=1}^{3}\left(m_{d_j}^2 I_{3d_j} V_{ud_j}V^\dagger_{d_j u}\right)\right]\\
C_+ &= -C_-
\end{aligned}
$$

$$
\begin{aligned}
\chi d\bar{d} \quad : \quad C_- &= -\frac{m_d}{4 m_W s_w}\left[\frac{(1+\lambda_{HV})\,Q_d^2 I_{3d}}{2c_w^2}-\frac{1}{32 s_w^2}\sum_{i=1}^{3}\left(V_{u_i d}V^\dagger_{du_i}\right)+\frac{I_{3d}}{c_w^2}\left(\frac{1}{32s_w^2}\right.\right.\\
&- \left.\left.\frac{(1+\lambda_{HV})\,Q_d I_{3d}}{2}\right)-\frac{1}{16 m_W^2 s_w^2}\sum_{i=1}^{3}\left(m_{u_i}^2 I_{3u_i} V_{u_i d}V^\dagger_{du_i}\right)\right]\\
C_+ &= -C_-
\end{aligned}
$$

$$
\begin{aligned}
\chi l\bar{l} \quad : \quad C_- &= -\frac{m_l}{4 m_W s_w}\left[\frac{(1+\lambda_{HV})\,Q_l^2 I_{3l}}{2c_w^2}-\frac{1}{32 s_w^2}+\frac{I_{3l}}{c_w^2}\left(\frac{1}{32s_w^2}-\frac{(1+\lambda_{HV})\,Q_l I_{3l}}{2}\right)\right.\\
&- \left.\frac{m_l^2 I_{3l}}{8 m_W^2 s_w^2}\left(-\frac{1}{4}+I_{3l}^2\right)\right]\\
C_+ &= -C_-
\end{aligned}
$$

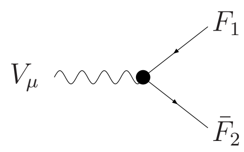
$$
\begin{aligned}
\chi\nu_l\bar{\nu}_l \quad : \quad C_- &= 0\\
C_+ &= 0
\end{aligned}
$$

$$
\begin{aligned}
\phi^- u\bar{d} \quad : \quad C_- &= -\frac{i m_d V^\dagger_{du}}{4\sqrt{2} m_W s_w}\left[\frac{1}{c_w^2}\left(\frac{-1}{16}-\frac{(1+\lambda_{HV})\,Q_u Q_d}{2}\right)-\frac{3}{32 s_w^2}\right.\\
&- \left.\frac{m_u^2}{16 s_w^2 m_W^2}+\frac{I_{3u}}{c_w^2}\left(\frac{(1+\lambda_{HV})\,Q_d}{2}+\frac{1}{16}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
C_+ &= \frac{im_u V^\dagger_{du}}{4\sqrt{2}m_W s_w}\left[\frac{1}{c_w^2}\left(\frac{-1}{16} - \frac{(1+\lambda_{HV})\,Q_u Q_d}{2}\right) - \frac{3}{32 s_w^2}\right.\\
&- \left.\frac{m_d^2}{16 s_w^2 m_W^2} + \frac{I_{3d}}{c_w^2}\left(\frac{(1+\lambda_{HV})\,Q_u}{2} - \frac{1}{16}\right)\right]\\
\phi^+ d\bar{u} \;:\; C_- &= -\frac{im_u V_{ud}}{4\sqrt{2}s_w m_W}\left[\frac{1}{c_w^2}\left(\frac{1}{16} + \frac{(1+\lambda_{HV})\,Q_u Q_d}{2}\right) + \frac{3}{32 s_w^2}\right.\\
&+ \left.\frac{m_d^2}{16 s_w^2 m_W^2} - \frac{I_{3d}}{c_w^2}\left(\frac{(1+\lambda_{HV})\,Q_u}{2} - \frac{1}{16}\right)\right]\\
C_+ &= \frac{im_d V_{ud}}{4\sqrt{2}m_W s_w}\left[\frac{1}{c_w^2}\left(\frac{1}{16} + \frac{(1+\lambda_{HV})\,Q_u Q_d}{2}\right) + \frac{3}{32 s_w^2}\right.\\
&+ \left.\frac{m_u^2}{16 s_w^2 m_W^2} - \frac{I_{3u}}{c_w^2}\left(\frac{(1+\lambda_{HV})\,Q_d}{2} + \frac{1}{16}\right)\right]\\
\phi^- \nu_l \bar{l} \;:\; C_- &= -\frac{im_l}{4\sqrt{2}m_W s_w}\left[\frac{Q_l}{16 c_w^2} - \frac{3}{32 s_w^2} + \frac{I_{3\nu_l}}{c_w^2}\left(\frac{(1+\lambda_{HV})\,Q_l}{2} + \frac{1}{16}\right)\right]\\
C_+ &= 0\\
\phi^+ l \bar{\nu}_l \;:\; C_- &= 0\\
C_+ &= \frac{im_l}{4\sqrt{2}m_W s_w}\left[-\frac{Q_l}{16 c_w^2} + \frac{3}{32 s_w^2} - \frac{I_{3\nu_l}}{c_w^2}\left(\frac{(1+\lambda_{HV})\,Q_l}{2} + \frac{1}{16}\right)\right] \qquad (4.5)
\end{aligned}
$$

4.2.5 Vector-Fermion-Fermion effective vertices

The generic effective vertex is



$$= \frac{ie^3}{\pi^2}(C_-\Omega^- + C_+\Omega^+)\gamma_\mu$$

with the actual values of V, F_1, $\bar{F}_2$, C_- and C_+

$$A u\bar{u} \;:\; C_- = \frac{1}{4}\left[\frac{(1+\lambda_{HV})\,Q_u^3}{4c_w^2} + \frac{m_u^2}{8 s_w^2 m_W^2}\left(\frac{1}{2}\sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_j u}Q_{d_j}\right)\right.\right.$$

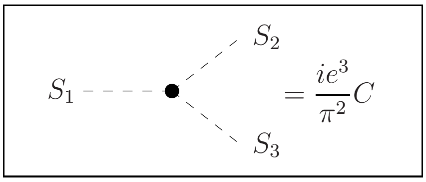
$$
\begin{aligned}
&& + & \quad \frac{Q_u}{4} + Q_u I_{3u}^2 \Big) \Big] \\
&C_+ & = & \frac{1}{4}\left[\frac{(1+\lambda_{HV})\,Q_u^3}{4c_w^2} - \frac{(1+\lambda_{HV})\,Q_u^2 I_{3u}}{2c_w^2} + \frac{(1+\lambda_{HV})\,Q_u I_{3u}^2}{4s_w^2 c_w^2}\right. \\
&&& + \frac{1}{4s_w^2}\left(\frac{1}{4m_W^2}\sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_j u} m_{d_j}^2 Q_{d_j}\right)\right. \\
&&& \left.\left. + \frac{m_u^2 Q_u\,(1+4I_{3u}^2)}{8m_W^2} + \sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_j u}\left(1+Q_{d_j}\right)\frac{(1+\lambda_{HV})}{2}\right)\right)\right] \\
Ad\bar{d} \;:\; & C_- & = & \frac{1}{4}\left[\frac{(1+\lambda_{HV})\,Q_d^3}{4c_w^2} + \frac{m_d^2}{8s_w^2 m_W^2}\left(\frac{1}{2}\sum_{i=1}^{3}\left(V_{u_i d}V^\dagger_{du_i} Q_{u_i}\right)\right.\right. \\
&& + & \quad \left.\left.\frac{Q_d}{4} + Q_d I_{3d}^2\right)\right] \\
&C_+ & = & \frac{1}{4}\left[\frac{(1+\lambda_{HV})\,Q_d^3}{4c_w^2} - \frac{(1+\lambda_{HV})\,Q_d^2 I_{3d}}{2c_w^2} + \frac{(1+\lambda_{HV})\,Q_d I_{3d}^2}{4s_w^2 c_w^2}\right. \\
&&& + \frac{1}{4s_w^2}\left(\frac{1}{4m_W^2}\sum_{i=1}^{3}\left(V_{u_i d}V^\dagger_{du_i} m_{u_i}^2 Q_{u_i}\right)\right. \\
&&& \left.\left. + \frac{m_d^2 Q_d\,(1+4I_{3d}^2)}{8m_W^2} + \sum_{i=1}^{3}\left(V_{u_i d}V^\dagger_{du_i}\,(Q_{u_i}-1)\,\frac{(1+\lambda_{HV})}{2}\right)\right)\right] \\
Al\bar{l} \;:\; & C_- & = & \frac{1}{4}\left[\frac{(1+\lambda_{HV})\,Q_l^3}{4c_w^2} + \frac{m_l^2}{8s_w^2 m_W^2}\left(\frac{Q_l}{4} + Q_l I_{3l}^2\right)\right] \\
&C_+ & = & \frac{1}{4}\left[\frac{(1+\lambda_{HV})\,Q_l^3}{4c_w^2} - \frac{(1+\lambda_{HV})\,Q_l^2 I_{3l}}{2c_w^2} + \frac{(1+\lambda_{HV})\,Q_l I_{3l}^2}{4s_w^2 c_w^2}\right. \\
&&& \left. + \frac{1}{4s_w^2}\left(\frac{m_l^2 Q_l\,(1+4I_{3l}^2)}{8m_W^2} - \frac{(1+\lambda_{HV})}{2}\right)\right] \\
A\nu_l\bar{\nu}_l \;:\; & C_- & = & 0 \\
&C_+ & = & \frac{1}{32s_w^2}\left[\frac{m_l^2 Q_l}{2m_W^2} + (Q_l+1)\,(1+\lambda_{HV})\right] \\
Zu\bar{u} \;:\; & C_- & = & \frac{1}{8c_w}\left\{\frac{(1+\lambda_{HV})\,Q_u^3 s_w}{2c_w^2} + \frac{m_u^2}{8s_w m_W^2}\left[\sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_j u}\left(Q_{d_j} - \frac{I_{3d_j}}{s_w^2}\right)\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
& & & + \left(Q_u - \frac{I_{3u}}{s_w^2}\right)\Bigg]\Bigg\} \\
& & C_+ &= \frac{1}{8c_w}\left\{\frac{(1+\lambda_{HV})\,Q_u^3 s_w}{2c_w^2} - \frac{(1+\lambda_{HV})\,Q_u^2 I_{3u}(1+2s_w^2)}{2s_w c_w^2}\right. \\
& & & +3\,\frac{(1+\lambda_{HV})\,Q_u I_{3u}^2}{2s_w c_w^2} - \frac{(1+\lambda_{HV})\,I_{3u}^3}{2s_w^3 c_w^2} \\
& & & + \frac{1}{2s_w}\left[\frac{1}{4m_W^2}\left(\sum_{j=1}^{3}\left(V_{ud_j}V^{\dagger}_{d_j u} m_{d_j}^2 Q_{d_j}\right) + \frac{m_u^2 Q_u\,(1+4I_{3u}^2)}{2}\right)\right. \\
& & & + \left.\left.\sum_{j=1}^{3}\left(V_{ud_j}V^{\dagger}_{d_j u}\frac{(1+\lambda_{HV})}{2}\left(Q_{d_j} - \frac{c_w^2 + I_{3d_j}}{s_w^2}\right)\right)\right]\right\} \\
Zd\bar{d} &: & C_- &= \frac{1}{8c_w}\left\{\frac{(1+\lambda_{HV})\,Q_d^3 s_w}{2c_w^2} + \frac{m_d^2}{8s_w m_W^2}\left[\sum_{i=1}^{3}\left(V_{u_i d}V^{\dagger}_{du_i}\left(Q_{u_i} - \frac{I_{3u_i}}{s_w^2}\right)\right)\right.\right. \\
& & & + \left.\left.\left(Q_d - \frac{I_{3d}}{s_w^2}\right)\right]\right\} \\
& & C_+ &= \frac{1}{16c_w}\left\{(1+\lambda_{HV})\left(\frac{Q_d^3 s_w}{c_w^2} - \frac{Q_d^2 I_{3d}(1+2s_w^2)}{s_w c_w^2} + 3\frac{Q_d I_{3d}^2}{s_w c_w^2} - \frac{I_{3d}^3}{s_w^3 c_w^2}\right)\right. \\
& & & + \frac{1}{s_w}\left[\frac{1}{4m_W^2}\left(\sum_{i=1}^{3}\left(V_{u_i d}V^{\dagger}_{du_i} m_{u_i}^2 Q_{u_i}\right) + \frac{m_d^2 Q_d\,(1+4I_{3d}^2)}{2}\right)\right. \\
& & & + \left.\left.\sum_{i=1}^{3}\left(\frac{1+\lambda_{HV}}{2}\right)\left(V_{u_i d}V^{\dagger}_{du_i}\left(Q_{u_i} + \frac{c_w^2 - I_{3u_i}}{s_w^2}\right)\right)\right]\right\} \\
Zl\bar{l} &: & C_- &= \frac{1}{8c_w}\left\{\frac{(1+\lambda_{HV})\,Q_l^3 s_w}{2c_w^2} + \frac{m_l^2}{4s_w m_W^2}\left[-\frac{1}{4s_w^2} + \frac{1}{2}\left(Q_l - \frac{I_{3l}}{s_w^2}\right)\right]\right\} \\
& & C_+ &= \frac{1}{16c_w}\left\{\left(\frac{Q_l^3 s_w}{c_w^2} - \frac{Q_l^2 I_{3l}(1+2s_w^2)}{s_w c_w^2}\right.\right. \\
& & & \left. +3\,\frac{Q_l I_{3l}^2}{s_w c_w^2} - \frac{I_{3l}^3}{s_w^3 c_w^2}\right)(1+\lambda_{HV}) + \frac{1}{2s_w}\left[\frac{m_l^2 Q_l}{2m_W^2}\right. \\
& & & + \left.\left.\frac{1}{s_w^2}\,(1+\lambda_{HV})\left(c_w^2 - I_{3\nu_l}\right)\right]\right\} \\
Z\nu_l\bar{\nu}_l &: & C_- &= 0
\end{aligned}
$$

$$
\begin{array}{rcrcl}
 & & C_+ & = & \frac{1}{16c_w}\left\{-\frac{(1+\lambda_{HV})\,I^3_{3\nu_l}}{s_w^3c_w^2}+\frac{1}{2s_w}\left[\frac{m_l^2Q_l}{2m_W^2}\right.\right. \\
 & & & & \left.\left.+\;(1+\lambda_{HV})\left(Q_l-\frac{c_w^2+I_{3l}}{s_w^2}\right)\right]\right\} \\
W^-u\bar{d} & : & C_- & = & 0 \\
 & & C_+ & = & \frac{V_{du}^\dagger}{16\sqrt{2}s_w}\left[\frac{Q_dI_{3u}+Q_uI_{3d}-Q_uQ_d}{c_w^2}-\frac{1}{s_w^2}+\frac{1}{4s_w^2c_w^2}\right](1+\lambda_{HV}) \\
W^+d\bar{u} & : & C_- & = & 0 \\
 & & C_+ & = & \frac{V_{ud}}{16\sqrt{2}s_w}\left[\frac{Q_dI_{3u}+Q_uI_{3d}-Q_uQ_d}{c_w^2}-\frac{1}{s_w^2}+\frac{1}{4s_w^2c_w^2}\right](1+\lambda_{HV}) \\
\left.\begin{array}{l} W^-\nu_l\bar{l} \\ W^+l\bar{\nu}_l \end{array}\right\} & : & C_- & = & 0 \\
 & & C_+ & = & \frac{1}{16\sqrt{2}s_w}\left[\frac{Q_lI_{3\nu_l}}{c_w^2}-\frac{1}{s_w^2}+\frac{1}{4s_w^2c_w^2}\right](1+\lambda_{HV})
\end{array}
\tag{4.6}
$$

4.2.6 Scalar-Scalar-Scalar effective vertices

The generic effective vertex is



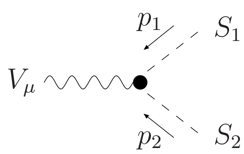
with the actual values of S_1, S_2, S_3, and C

$$
\begin{array}{rcrcl}
\left.\begin{array}{l} HH\chi \\ \chi\chi\chi \\ \chi\phi^+\phi^- \end{array}\right\} & : & C & = & 0 \\
HHH & : & C & = & \frac{3}{32s_w^3}\left[\frac{1-4\lambda_{HV}}{2}m_W+\frac{1}{m_W^3}\left(\sum_{i=1}^{6}m_{l_i}^4+N_{\rm col}\sum_{i=1}^{6}m_{q_i}^4\right)\right.
\end{array}
$$

$$
\begin{aligned}
& & & + \frac{1}{4}\left(1+\frac{1}{2c_w^2}\right)\frac{m_H^2}{m_W} + \frac{(1-4\lambda_{HV})\,m_W}{4c_w^4}\Bigg] \\
H\chi\chi &: & C &= \frac{1}{8s_w^3}\left[\frac{1-4\lambda_{HV}}{8}m_W + \frac{1}{4m_W^3}\left(\sum_{i=1}^{6} m_{l_i}^4 + N_{\rm col}\sum_{i=1}^{6} m_{q_i}^4\right)\right. \\
& & & + \left.\frac{1}{16}\left(1+\frac{1}{2c_w^2}\right)\frac{m_H^2}{m_W} + \frac{(1-4\lambda_{HV})\,m_W}{16c_w^4}\right] \\
H\phi^+\phi^- &: & C &= \frac{1}{32s_w^3}\left[\frac{1}{m_W^3}\left(\sum_{i=1}^{3} m_{e_i}^4 + N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_i d_j}V^\dagger_{d_j u_i}(m_{u_i}^4 + m_{d_j}^4)\right)\right)\right. \\
& & & + \left.\frac{(1+2c_w^2)}{8c_w^2}\frac{m_H^2}{m_W} + \frac{3\,(1-4\lambda_{HV})}{4}m_W + \frac{1-4\lambda_{HV}}{4}\frac{s_w^2\,(1+c_w^2)}{c_w^4}m_W\right]
\end{aligned}
\tag{4.7}
$$

4.2.7 Vector-Scalar-Scalar effective vertices

The generic effective vertex is



$$= \frac{e^3}{\pi^2}C(p_1-p_2)_\mu$$

with the actual values of V, S_1, S_2, and C

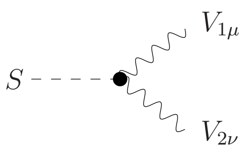
$$
\left.\begin{array}{l} AHH \\ ZHH \\ A\chi\chi \\ Z\chi\chi \end{array}\right\} : \quad C = 0
$$

$$
\begin{aligned}
A\chi H &: & C &= \frac{5}{192 s_w^2} \\
Z\chi H &: & C &= -\frac{1}{96 s_w c_w}\left[\frac{1+2c_w^2+20c_w^4}{8s_w^2c_w^2} + \frac{1}{s_w^2 m_W^2}\left(\sum_{i=1}^{6}\left(m_{l_i}^2 + N_{\rm col}m_{q_i}^2\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
A\phi^+\phi^- : \quad C &= \frac{i}{48 s_w^2}\left[\frac{1+12c_w^2}{8c_w^2} + \frac{1}{m_W^2}\left(-\sum_{i=1}^{3}\left(m_{e_i}^2 Q_{e_i}\right)\right.\right.\\
&+ \left.\left. N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_i d_j}V_{d_j u_i}^\dagger (m_{u_i}^2 + m_{d_j}^2)\right)\right)\right]\\
Z\phi^+\phi^- : \quad C &= \frac{i}{48 s_w c_w}\left\{\frac{1-24c_w^4}{16 c_w^2 s_w^2} + \frac{1}{m_W^2}\left(-\sum_{i=1}^{3}\left(m_{e_i}^2\left(Q_{e_i} + \frac{I_{3\nu_i}}{s_w^2}\right)\right)\right.\right.\\
&+ \left.\left. N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_i d_j}V_{d_j u_i}^\dagger\left[(m_{u_i}^2 + m_{d_j}^2) + \frac{m_{u_i}^2 I_{3d_j} - m_{d_i}^2 I_{3u_i}}{s_w^2}\right]\right)\right)\right\}\\
\left.\begin{array}{l} W^+\phi^- H \\ W^- H\phi^+ \end{array}\right\} : \quad C &= \frac{i}{96 s_w^3}\left[\frac{1+22c_w^2}{8c_w^2} + \frac{1}{m_W^2}\left(\sum_{i=1}^{3} m_{e_i}^2\right.\right.\\
&+ \left.\left. N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_i d_j}V_{d_j u_i}^\dagger (m_{u_i}^2 + m_{d_j}^2)\right)\right)\right]\\
\left.\begin{array}{l} W^+\phi^- \chi \\ W^- \phi^+\chi \end{array}\right\} : \quad C &= \frac{1}{48 s_w^3}\left[-\frac{1+22c_w^2}{16c_w^2} + \frac{1}{m_W^2}\left(\sum_{i=1}^{3}\left(m_{e_i}^2 I_{3e_i}\right)\right.\right.\\
&- \left.\left. N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_i d_j}V_{d_j u_i}^\dagger (m_{u_i}^2 I_{3u_i} - m_{d_j}^2 I_{3d_j})\right)\right)\right]
\end{aligned}
\tag{4.8}
$$

Scalar-Vector-Vector effective vertices

The generic effective vertex is



$$= \frac{ie^3}{\pi^2} C g_{\mu\nu}$$

with the actual values of S, V_1, V_2 and C

$$
\left.\begin{array}{l} \chi AA \\ \chi AZ \\ \chi ZZ \\ \chi W^- W^+ \end{array}\right\} : \quad C = 0
$$

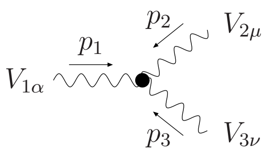
$$
\begin{aligned}
HAA \ &: \ C \ = \ -\frac{1}{8s_w}\left[\frac{1}{m_W}\left(\sum_{i=1}^{6}\left(m_{l_i}^2 Q_{l_i}^2\right) + N_{\rm col}\sum_{i=1}^{6}\left(m_{q_i}^2 Q_{q_i}^2\right)\right) + \frac{m_W}{2}\right] \\
HAZ \ &: \ C \ = \ \frac{1}{8c_w}\left\{\frac{1}{m_W}\left[\sum_{i=1}^{6}\left(m_{l_i}^2 Q_{l_i}\left(\frac{I_{3l_i}}{2s_w^2} - Q_{l_i}\right)\right)\right.\right. \\
&\qquad + \ N_{\rm col}\sum_{i=1}^{6}\left.\left(m_{q_i}^2 Q_{q_i}\left(\frac{I_{3q_i}}{2s_w^2} - Q_{q_i}\right)\right)\right] + \left.\frac{m_W\left(1+2c_w^2\right)}{4s_w^2}\right\} \\
HZZ \ &: \ C \ = \ \frac{1}{8}\left\{\frac{1}{m_W c_w^2}\left[\sum_{i=1}^{6}\left(m_{l_i}^2\left(\frac{Q_{l_i} I_{3l_i}}{s_w} - Q_{l_i}^2 s_w - \frac{I_{3l_i}^2}{s_w^3}\right)\right)\right.\right. \\
&\qquad + \ N_{\rm col}\sum_{i=1}^{6}\left.\left(m_{q_i}^2\left(\frac{Q_{q_i} I_{3q_i}}{s_w} - Q_{q_i}^2 s_w - \frac{I_{3q_i}^2}{s_w^3}\right)\right)\right] + \left.\frac{m_W\left(s_w^2-2\right)}{2s_w^3}\right\} \\
HW^-W^+ \ &: \ C \ = \ -\frac{1}{8s_w^3}\left[\frac{1}{4m_W}\left(\sum_{i=1}^{3} m_{e_i}^2\right.\right. \\
&\qquad + \ N_{\rm col}\sum_{i,j=1}^{3}\left.\left.\left(V_{u_i d_j}V_{d_j u_i}^{\dagger}\left(m_{u_i}^2 + m_{d_j}^2\right)\right)\right) + m_W\right] \\
\left.\begin{matrix}\phi^- A W^+ \\ \phi^+ W^- A\end{matrix}\right\} &: \ C \ = \ \frac{1}{32 s_w^2} K \\
\left.\begin{matrix}\phi^- Z W^+ \\ \phi^+ W^- Z\end{matrix}\right\} &: \ C \ = \ \frac{1}{32 s_w c_w} K
\end{aligned}
\tag{4.9}
$$

where

$$
K \ = \ m_W + \frac{N_{\rm col}}{m_W}\sum_{i,j=1}^{3}\left(V_{u_i d_j}V_{d_j u_i}^{\dagger}\left(Q_{u_i} m_{d_j}^2 - Q_{d_j} m_{u_i}^2\right)\right) \tag{4.10}
$$

4.2.8 Vector-Vector-Vector effective vertices

The generic effective vertex is



$$= \frac{ie^3}{\pi^2} C \left[g_{\alpha\mu}(p_2 - p_1)_\nu + g_{\mu\nu}(p_3 - p_2)_\alpha + g_{\nu\alpha}(p_1 - p_3)_\mu \right]$$

with the actual values of V_1, V_2, V_3 and C

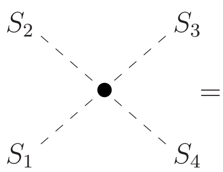
$$\begin{array}{rcl} \left.\begin{array}{l} AAA \\ AAZ \\ AZZ \\ ZZZ \end{array}\right\} : \quad C &=& 0 \\ AW^+W^- \quad : \quad C &=& K \\ ZW^+W^- \quad : \quad C &=& -\dfrac{c_w}{s_w} K \end{array} \tag{4.11}$$

where

$$K = \frac{7 + 4\lambda_{HV}}{96 s_w^2} + \frac{1}{48 s_w^2} \left[\sum_{i=1}^{3} 1 + N_{\mathrm{col}} \sum_{i,j=1}^{3} \left(V_{u_i d_j} V^\dagger_{d_j u_i} \right) \right] \tag{4.12}$$

4.2.9 Scalar-Scalar-Scalar-Scalar effective vertices

The generic effective vertex is



$$= \frac{ie^4}{\pi^2} C$$

with the actual values of S_1, S_2, S_3, S_4 and C

$$\left.\begin{array}{l} HHH\chi \\ H\chi\chi\chi \\ H\chi\phi^-\phi^+ \end{array}\right\} : \quad C = 0$$

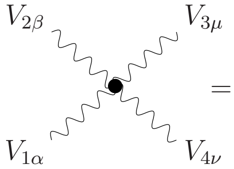
$$
\begin{aligned}
\left.\begin{array}{r} HHHH \\ \chi\chi\chi\chi \end{array}\right\} : \quad C &= \frac{1}{64 s_w^4} K_1 \\
HH\chi\chi \quad : \quad C &= \frac{1}{192 s_w^4} K_1 \\
\left.\begin{array}{r} HH\phi^-\phi^+ \\ \chi\chi\phi^-\phi^+ \end{array}\right\} : \quad C &= \frac{1}{64 s_w^4} K_2 \\
\phi^-\phi^+\phi^-\phi^+ \quad : \quad C &= \frac{1}{32 s_w^4} K_3
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
K_1 &= \frac{1}{m_W^2}\left[\frac{5}{m_W^2}\sum_{i=1}^{6}\left(m_{l_i}^4 + N_{\text{col}} m_{q_i}^4\right) + \frac{3}{2} m_H^2 \left(1 + \frac{1}{2c_w^2}\right)\right] + \frac{1-12\lambda_{HV}}{2}\left(1 + \frac{1}{2c_w^4}\right) \\
K_2 &= \frac{1}{m_W^2}\left[\frac{5}{3m_W^2}\left(\sum_{i=1}^{3} m_{e_i}^4 + N_{\text{col}} \sum_{i,j=1}^{3} V_{u_i d_j} V^\dagger_{d_j u_i}\left(m_{u_i}^4 + m_{d_j}^4\right)\right) + \frac{1}{2} m_H^2 \left(1 + \frac{1}{2c_w^2}\right)\right] \\
&+ \frac{1-12\lambda_{HV}}{4}\left(1 + \frac{s_w^2}{3c_w^2}\left(1 + \frac{1}{c_w^2}\right)\right) \\
K_3 &= \frac{1}{m_W^2}\left[\frac{5}{3m_W^2}\left(\sum_{i=1}^{3} m_{e_i}^4 + N_{\text{col}} \sum_{i,j,k,l=1}^{3}\left(V_{u_i d_j} V^\dagger_{d_j u_k} V_{u_k d_l} V^\dagger_{d_l u_i}\left(m_{u_i}^2 m_{u_k}^2 + m_{d_j}^2 m_{d_l}^2\right)\right)\right)\right. \\
&+ \left.\frac{1}{2} m_h^2 \left(1 + \frac{1}{2c_w^2}\right)\right] + \left(\left(\frac{1}{4} - 3\lambda_{HV}\right)\left(1 + s_w^4\right)\right. \\
&+ \left.\left(\frac{1}{6} - 2\lambda_{HV}\right)\left(s_w^2 + \frac{2s_w^6}{c_w^2}\right) + \left(\frac{1}{12} - \lambda_{HV}\right)\frac{s_w^8}{c_w^4}\right)
\end{aligned}
\tag{4.14}
$$

4.2.10 Vector-Vector-Vector-Vector effective vertices

The generic effective vertex is



$$= \frac{ie^4}{\pi^2}[C_1 g_{\alpha\beta}g_{\mu\nu} + C_2 g_{\alpha\mu}g_{\beta\nu} + C_3 g_{\alpha\nu}g_{\beta\mu}]$$

with the actual values of V_1, V_2, V_3, V_4 C_1, C_2 and C_3

$$\begin{aligned}
AAAA \;:\; C_1 &= \frac{1}{12}\left(-1 + \sum_{i=1}^{6} Q_{l_i}^4 + N_{\rm col}\sum_{i=1}^{6} Q_{q_i}^4\right)\\
C_2 &= C_1\\
C_3 &= C_1
\end{aligned}$$

$$\begin{aligned}
AAAZ \;:\; C_1 &= \frac{1}{12}\left[\frac{c_w}{s_w} + \sum_{i=1}^{6}\left(\frac{s_w}{c_w}Q_{l_i}^4 - \frac{1}{2s_wc_w}Q_{l_i}^3 I_{3l_i}\right)\right.\\
&+ \left. N_{\rm col}\sum_{i=1}^{6}\left(\frac{s_w}{c_w}Q_{q_i}^4 - \frac{1}{2s_wc_w}Q_{q_i}^3 I_{3q_i}\right)\right]\\
C_2 &= C_1\\
C_3 &= C_1
\end{aligned}$$

$$\begin{aligned}
AAZZ \;:\; C_1 &= \frac{1}{12}\left[-\frac{c_w^2}{s_w^2} + \frac{1}{2}\sum_{i=1}^{6}\left(\frac{s_w^2}{c_w^2}Q_{l_i}^4 + \left(\frac{s_w}{c_w}Q_{l_i}^2 - \frac{1}{s_wc_w}Q_{l_i}I_{3l_i}\right)^2\right)\right.\\
&+ \left. \frac{N_{\rm col}}{2}\sum_{i=1}^{6}\left(\frac{s_w^2}{c_w^2}Q_{q_i}^4 + \left(\frac{s_w}{c_w}Q_{q_i}^2 - \frac{1}{s_wc_w}Q_{q_i}I_{3q_i}\right)^2\right)\right]\\
C_2 &= C_1\\
C_3 &= C_1
\end{aligned}$$

$$\begin{aligned}
AZZZ \;:\; C_1 &= \frac{1}{12}\left[\frac{c_w^3}{s_w^3} + \sum_{i=1}^{6}\left(\frac{s_w^3}{c_w^3}Q_{l_i}^4 - \frac{3}{2}\frac{s_w}{c_w^3}Q_{l_i}^3 I_{3l_i}\right.\right.\\
&+ \left. \frac{3}{2}\frac{1}{s_wc_w^3}Q_{l_i}^2 I_{3l_i}^2 - \frac{1}{2s_w^3c_w^3}Q_{l_i}I_{3l_i}^3\right)\\
&+ N_{\rm col}\sum_{i=1}^{6}\left(\frac{s_w^3}{c_w^3}Q_{q_i}^4 - \frac{3}{2}\frac{s_w}{c_w^3}Q_{q_i}^3 I_{3q_i}\right.
\end{aligned}$$

$$
\begin{aligned}
&& &+ \frac{3}{2}\frac{1}{s_w c_w^3} Q_{q_i}^2 I_{3q_i}^2 - \frac{1}{2s_w^3 c_w^3} Q_{q_i} I_{3q_i}^3 \Bigg)\Bigg] \\
&& C_2 &= C_1 \\
&& C_3 &= C_1
\end{aligned}
$$

$$
\begin{aligned}
ZZZZ &: & C_1 &= \frac{1}{12}\left[-\frac{c_w^4}{s_w^4} + \sum_{i=1}^{6}\left(\frac{s_w^4}{c_w^4} Q_{l_i}^4 - 2\frac{s_w^2}{c_w^4} Q_{l_i}^3 I_{3l_i} \right.\right. \\
&& &+ \left. \frac{3}{c_w^4} Q_{l_i}^2 I_{3l_i}^2 - \frac{2}{s_w^2 c_w^4} Q_{l_i} I_{3l_i}^3 + \frac{1}{2s_w^4 c_w^4} I_{3l_i}^4 \right) \\
&& &+ N_{\rm col} \sum_{i=1}^{6}\left(\frac{s_w^4}{c_w^4} Q_{q_i}^4 - 2\frac{s_w^2}{c_w^4} Q_{q_i}^3 I_{3q_i} \right. \\
&& &+ \left.\left. \frac{3}{c_w^4} Q_{q_i}^2 I_{3q_i}^2 - \frac{2}{s_w^2 c_w^4} Q_{q_i} I_{3q_i}^3 + \frac{1}{2s_w^4 c_w^4} I_{3q_i}^4 \right)\right] \\
&& C_2 &= C_1 \\
&& C_3 &= C_1
\end{aligned}
$$

$$
\begin{aligned}
AAW^-W^+ &: & C_1 &= \frac{1}{16 s_w^2}\left[\frac{10+4\lambda_{HV}}{3} + \sum_{i=1}^{3} 1 + \frac{25}{27} N_{\rm col} \sum_{i,j=1}^{3} \left(V_{u_i d_j} V_{d_j u_i}^\dagger \right) \right] \\
&& C_2 &= -\frac{1}{16 s_w^2}\left[\frac{7+2\lambda_{HV}}{3} + \frac{1}{3}\sum_{i=1}^{3} 1 + \frac{11}{27} N_{\rm col} \sum_{i,j=1}^{3} \left(V_{u_i d_j} V_{d_j u_i}^\dagger \right) \right] \\
&& C_3 &= C_2
\end{aligned}
$$

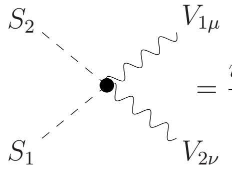
$$
\begin{aligned}
AZW^-W^+ &: & C_1 &= \frac{1}{16 s_w c_w}\left[-\frac{(10+4\lambda_{HV})\, c_w^2}{3 s_w^2} + \left(1 - \frac{11}{12 s_w^2} \right) \sum_{i=1}^{3} 1 \right. \\
&& &+ \left. N_{\rm col} \sum_{i,j=1}^{3} \left(V_{u_i d_j} V_{d_j u_i}^\dagger \left(\frac{25}{27} - \frac{11}{12 s_w^2} \right) \right) \right] \\
&& C_2 &= \frac{1}{16 s_w c_w}\left[\frac{7+2\lambda_{HV}}{3} \frac{c_w^2}{s_w^2} + \left(\frac{5}{12 s_w^2} - \frac{1}{3} \right) \sum_{i=1}^{3} 1 \right. \\
&& &+ \left. N_{\rm col} \sum_{i,j=1}^{3} \left(V_{u_i d_j} V_{d_j u_i}^\dagger \left(\frac{5}{12 s_w^2} - \frac{11}{27} \right) \right) \right] \\
&& C_3 &= C_2
\end{aligned}
$$

$$
\begin{aligned}
ZZW^-W^+ \quad : \quad C_1 &= \frac{(5+2\lambda_{HV})\,c_w^2}{24s_w^4} + \frac{1}{16c_w^2}\left[\left(1 - \frac{11}{6s_w^2} + \frac{11}{12s_w^4}\right)\sum_{i=1}^{3} 1\right. \\
&+ \left. N_{\rm col}\left(\frac{25}{27} - \frac{11}{6s_w^2} + \frac{11}{12s_w^4}\right)\sum_{i,j=1}^{3}\left(V_{u_i d_j}V^\dagger_{d_j u_i}\right)\right] \\
C_2 &= -\frac{(7+2\lambda_{HV})\,c_w^2}{48s_w^4} + \frac{1}{16c_w^2}\left[\left(-\frac{1}{3} + \frac{5}{6s_w^2} - \frac{5}{12s_w^4}\right)\sum_{i=1}^{3} 1\right. \\
&+ \left. N_{\rm col}\left(-\frac{11}{27} + \frac{5}{6s_w^2} - \frac{5}{12s_w^4}\right)\sum_{i,j=1}^{3}\left(V_{u_i d_j}V^\dagger_{d_j u_i}\right)\right] \\
C_3 &= C_2
\end{aligned}
$$

$$
\begin{aligned}
W^-W^+W^-W^+ \quad : \quad C_1 &= \frac{1}{16s_w^4}\left[\frac{3+2\lambda_{HV}}{3} + \frac{1}{2}\sum_{i=1}^{3} 1\right. \\
&+ \left. \frac{N_{\rm col}}{2}\sum_{i,j,k,m=1}^{3}\left(V_{u_i d_j}V^\dagger_{d_j u_k}V_{u_k d_m}V^\dagger_{d_m u_i}\right)\right] \\
C_2 &= -\frac{1}{8s_w^4}\left[\frac{7+2\lambda_{HV}}{3} + \frac{5}{12}\sum_{i=1}^{3} 1\right. \\
&+ \left. \frac{5}{12}N_{\rm col}\sum_{i,j,k,m=1}^{3}\left(V_{u_i d_j}V^\dagger_{d_j u_k}V_{u_k d_m}V^\dagger_{d_m u_i}\right)\right] \\
C_3 &= C_1
\end{aligned} \tag{4.15}
$$

4.2.11 Scalar-Scalar-Vector-Vector effective vertices

The generic effective vertex is



$$= \frac{ie^4}{\pi^2} C g_{\mu\nu}$$

with the actual values of S_1, S_2, V_1, V_2 and C

$$
\left.\begin{array}{l} H\chi AA \\ H\chi AZ \\ H\chi ZZ \\ H\chi W^+W^- \end{array}\right\}: \; C \; = \; 0
$$

$$
\left.\begin{array}{l} HHAA \\ \chi\chi AA \end{array}\right\}: \; C \; = \; \frac{1}{16s_w^2}\left\{\frac{1}{12} - \frac{1}{m_W^2}\left[\sum_{i=1}^{6}\left(Q_{l_i}^2 m_{l_i}^2\right) + N_{\rm col}\sum_{i=1}^{6}\left(Q_{q_i}^2 m_{q_i}^2\right)\right]\right\}
$$

$$
\begin{aligned}
\left.\begin{array}{l} HHAZ \\ \chi\chi AZ \end{array}\right\}: \; C \; &= \; \frac{1}{16s_w}\left\{\frac{4+s_w^2}{12s_w^2c_w} + \frac{1}{m_W^2c_w}\left[\sum_{i=1}^{6}\left(Q_{l_i}m_{l_i}^2\left(\frac{I_{3l_i}}{2s_w^2} - Q_{l_i}\right)\right)\right.\right. \\
&+ \; \left.\left. N_{\rm col}\sum_{i=1}^{6}\left(Q_{q_i}m_{q_i}^2\left(\frac{I_{3q_i}}{2s_w^2} - Q_{q_i}\right)\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left.\begin{array}{l} HHZZ \\ \chi\chi ZZ \end{array}\right\}: \; C \; &= \; -\frac{1}{16c_w^2}\left\{\frac{1+2c_w^2+40c_w^4-4c_w^6}{48s_w^4c_w^2}\right. \\
&+ \; \frac{1}{m_W^2}\left[\sum_{i=1}^{6}\left(m_{l_i}^2\left(Q_{l_i}^2 + \frac{4I_{3l_i}^2}{3s_w^4} - \frac{Q_{l_i}I_{3l_i}}{s_w^2}\right)\right)\right. \\
&+ \; \left.\left. N_{\rm col}\sum_{i=1}^{6}\left(m_{q_i}^2\left(Q_{q_i}^2 + \frac{4I_{3q_i}^2}{3s_w^4} - \frac{Q_{q_i}I_{3q_i}}{s_w^2}\right)\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
\left.\begin{array}{l} HHW^-W^+ \\ \chi\chi W^-W^+ \end{array}\right\}: \; C \; &= \; -\frac{1}{48s_w^4}\left\{\frac{1+38c_w^2}{16c_w^2}\right. \\
&+ \; \left.\frac{1}{m_W^2}\left[\sum_{i=1}^{3}m_{e_i}^2 + N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_id_j}V_{d_ju_i}^\dagger\left(m_{u_i}^2 + m_{d_j}^2\right)\right)\right]\right\}
\end{aligned}
$$

$$
\left.\begin{array}{l} H\phi^+W^-A \\ \phi^-HAW^+ \end{array}\right\}: \; C \; = \; K_1
$$

$$
\chi\phi^+W^-A \; : \; C \; = \; -iK_1
$$

$$
\phi^-\chi AW^+ \; : \; C \; = \; iK_1
$$

$$
\left.\begin{array}{l} H\phi^+W^-Z \\ \phi^-HZW^+ \end{array}\right\}: \; C \; = \; K_2
$$

$$\chi\phi^+W^-Z \quad : \quad C \quad = \quad -iK_2$$

$$\phi^-\chi Z W^+ \quad : \quad C \quad = \quad iK_2$$

$$\begin{aligned}
\phi^-\phi^+AA \quad : \quad C \quad &= \quad -\frac{1}{12s_w^2}\left\{\frac{1+21c_w^2}{16c_w^2}+\frac{1}{m_W^2}\left[\sum_{i=1}^{3}m_{e_i}^2\right.\right. \\
&+ \quad \left.\left.\frac{5}{6}N_{\rm col}\sum_{i,j=1}^{3}\left(V_{u_id_j}V_{d_ju_i}^\dagger(m_{u_i}^2+m_{d_j}^2)\right)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
\phi^-\phi^+AZ \quad : \quad C \quad &= \quad \frac{1}{12s_wc_w}\left\{\frac{42c_w^4-10c_w^2-1}{32s_w^2c_w^2}\right. \\
&- \quad \frac{1}{m_W^2}\left[\sum_{i=1}^{3}\left(m_{e_i}^2Q_{e_i}\left(Q_{e_i}+\frac{5}{8}\frac{I_{3\nu_i}}{s_w^2}\right)\right)\right. \\
&+ \quad N_{\rm col}\sum_{i,j=1}^{3}\left[V_{u_id_j}V_{d_ju_i}^\dagger\left(m_{u_i}^2\left(\frac{5}{6}-\frac{I_{3d_i}}{s_w^2}\left(Q_{d_j}-\frac{5}{8}Q_{u_i}\right)\right)\right.\right. \\
&+ \quad \left.\left.\left.\left.m_{d_j}^2\left(\frac{5}{6}-\frac{I_{3u_i}}{s_w^2}\left(Q_{u_i}-\frac{5}{8}Q_{d_j}\right)\right)\right)\right]\right]\right\}
\end{aligned}$$

$$\begin{aligned}
\phi^-\phi^+ZZ \quad : \quad C \quad &= \quad \frac{1}{12c_w^2}\left\{\frac{-1+2c_w^2+44c_w^4-84c_w^6}{64s_w^4c_w^2}\right. \\
&- \quad \frac{1}{m_W^2}\left[\sum_{i=1}^{3}\left(m_{e_i}^2\left(Q_{e_i}^2+\frac{5}{4}\frac{Q_{e_i}I_{3\nu_i}}{s_w^2}+\frac{I_{3\nu_i}^2}{s_w^4}\right)\right)\right. \\
&+ \quad N_{\rm col}\sum_{i,j=1}^{3}\left[V_{u_id_j}V_{d_ju_i}^\dagger\left(m_{u_i}^2\left(\frac{5}{6}-\frac{I_{3d_i}}{s_w^2}\left(2Q_{d_j}-\frac{5}{4}Q_{u_i}\right)+\frac{I_{3d_i}^2}{s_w^4}\right)\right.\right. \\
&+ \quad \left.\left.\left.\left.m_{d_j}^2\left(\frac{5}{6}-\frac{I_{3u_i}}{s_w^2}\left(2Q_{u_i}-\frac{5}{4}Q_{d_j}\right)+\frac{I_{3u_i}^2}{s_w^4}\right)\right]\right]\right\}
\end{aligned}$$

$$\begin{aligned}
\phi^-\phi^+W^-W^+ \quad : \quad C \quad &= \quad -\frac{1}{48s_w^4}\left\{\frac{1}{m_W^2}\left[\left(\sum_{i=1}^{3}m_{e_i}^2\right.\right.\right. \\
&+ \quad \left.\left.N_{\rm col}\sum_{i,j,k,l=1}^{3}\left(V_{u_id_j}V_{d_ju_k}^\dagger V_{u_kd_l}V_{d_lu_i}^\dagger\left(m_{u_i}m_{u_k}+m_{d_j}m_{d_l}\right)\right)\right)\right] \\
&+ \quad \left.\frac{38c_w^2+1}{16c_w^2}\right\}
\end{aligned}$$

(4.16)

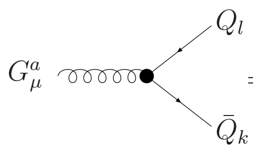
with

$$
\begin{aligned}
K_1 &= \frac{1}{24s_w^3}\left\{\frac{1+22c_w^2}{32c_w^2}+K\right\} \\
K_2 &= \frac{1}{24s_w^2c_w}\left\{\frac{1+21c_w^2-22c_w^4}{32c_w^2s_w^2}+K\right\} \\
K &= \frac{1}{8m_W^2}\left[\sum_{i=1}^{3}m_{e_i}^2+N_{\mathrm{col}}\sum_{i,j=1}^{3}\left(V_{u_id_j}V^\dagger_{d_ju_i}\left(3m_{d_j}^2+2m_{u_i}^2\right)\right)\right] \quad (4.17)
\end{aligned}
$$

4.2.12 Mixed Electroweak/QCD corrections

In [21], all mixed R_2 QCD/Electroweak vertices with internal QCD particle and external weak fields are presented. For completeness, we give here the only contributing Mixed Electroweak/QCD R_2 effective vertex, with internal EW particles and external colored states.

The generic Gluon-Quark-Quark effective vertex is



$$= \frac{ig_se^2}{\pi^2}t^a_{kl}(C_-\Omega^-+C_+\Omega^+)\gamma_\mu$$

with the actual values of Q, $\bar{Q}$, C_- and C_+

$$
\begin{aligned}
u\bar{u} \;:\; C_- &= \frac{1}{16}\left[(1+\lambda_{HV})\frac{Q_u^2}{c_w^2}+\frac{m_u^2}{2s_w^2m_W^2}\left(\frac{1}{2}\sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_ju}\right)+\frac{1}{4}+I_{3u}^2\right)\right] \\
C_+ &= \frac{1}{16}\left[(1+\lambda_{HV})\left(\frac{1}{c_w^2}\left(Q_u^2+\frac{I_{3u}^2}{s_w^2}-2Q_uI_{3u}\right)+\frac{1}{2s_w^2}\sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_ju}\right)\right)\right. \\
&\quad \left.+\frac{1}{2m_W^2s_w^2}\left(\frac{1}{2}\sum_{j=1}^{3}\left(V_{ud_j}V^\dagger_{d_ju}m_{d_j}^2\right)+m_u^2\left(\frac{1}{4}+I_{3u}^2\right)\right)\right] \\
d\bar{d} \;:\; C_- &= \frac{1}{16}\left[(1+\lambda_{HV})\frac{Q_d^2}{c_w^2}+\frac{m_d^2}{2s_w^2m_W^2}\left(\frac{1}{2}\sum_{i=1}^{3}\left(V_{u_id}V^\dagger_{du_i}\right)+\frac{1}{4}+I_{3d}^2\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
C_+ \;=\; & \frac{1}{16}\left[(1+\lambda_{HV})\left(\frac{1}{c_w^2}\left(Q_d^2+\frac{I_{3d}^2}{s_w^2}-2Q_dI_{3d}\right)+\frac{1}{2s_w^2}\sum_{i=1}^{3}\left(V_{u_id}V_{du_i}^\dagger\right)\right)\right.\\
& \left.+\frac{1}{2m_W^2s_w^2}\left(\frac{1}{2}\sum_{i=1}^{3}\left(V_{u_id}V_{du_i}^\dagger m_{u_i}^2\right)+m_d^2\left(\frac{1}{4}+I_{3d}^2\right)\right)\right] \qquad (4.18)
\end{aligned}
$$

4.3 Tests of the calculation

We performed several checks on our formulae. First of all, we derived them by means of two independent calculations, secondly, we explicitly checked the gauge invariance of our results with the help of the Ward Identities listed in app. C, that we derived, by using the Background Field Method described in [34], in the way we detail in the appendix. Given the fact that only $R = R_1 + R_2$ is gauge invariant, we adopted the following strategy. The terms proportional to λ_{HV} in our effective vertices are expected to be gauge invariant by themselves. Such terms can only be generated by R_2, so that we could explicitly check, by using `FORM`, that this part of our results fulfills all of the Ward identities of app. C, both in the 't Hooft-Feynman gauge and in the Background Field Method approach. This provides an explicit test of the gauge invariance of the Four Dimensional Helicity regularization scheme in the complete Standard Model at 1-loop, and we consider this result as a by-product of our calculation.

To also test the parts not proportional to λ_{HV}, we computed analytically R_1 [3], we added it to R_2 and checked that the quantity $R_1 + R_2$ fulfills all of the 2-point and 3-point Ward identities listed in the appendix. In the 4-point case, many new vertices are present in R_1 that do not contribute to R_2, such as VVVS, and, given the fact that, after all, we just need to check R_2, we limited ourselves to verify only those Ward identities given in app. C, which include both the VVVV and VVV vertices, but not VVVS. The described gauge invariance test on $R_1 + R_2$ is a very powerful and non trivial one. In fact, the analytic expressions for R_1 are, in general, much more complicated than the ones for R_2, involving a huge amount of terms with different combinations/powers of Gram determinants.

As a final verification,an extra independent calculation appeared in the literature [35] that produced the same effective vertices using the same and

[3] We extracted the R_1 part of the contributing tensor integrals by using the Passarino-Veltman reduction technique and by further checking numerically the expressions with the help of CutTools [17].

different schemes. The authors compared their results with ours and came to the conclusion that they agree,proving once more the validity of the results.

Chapter 5

On the gauge invariance of the rational terms

As we know the result at the end of any calculation should be gauge invariant. It is also known that the C.C. part of the amplitude is also a gauge invariant quantity. As a consequence,the rational part of the one-loop amplitude,$R_1 + R_2$ is gauge invariant. Notice that seperately the two pieces do not have to be gauge invariant. In [22] it was argued that since there is a connection of the C.C. part and the R_1 part (the R_1 part can be fully recostructed from the 4-dimensional, gauge invariant, C.C. part) maybe gauge invariant properties are transfered to the R_1 part and therefore to the R_2 part as well, for physical processes.

This hypothesis was investigated in [23] and it turned out that it is not true. The rational part is only gauge invariant when both pieces of it are taken into account. However, calculating the effective vertices in different gauges can be also of some interest since it completes the theoretical picture on R_2 and allows tree level packages based on gauges different than 't Hooft-Feynman's to be transformed into one-loop calculators with the help of the techniques we have already described. In addition, the use of a general renormalizable R_ξ gauge can verify the numerical stability of one-loop predictions by studying invariance of the results for different values of ξ.

5.1 R_ξ and Unitary gauges

For the calculation in R_ξ and Unitary gauges we have to change some Feynman rules. Feynman rules containing ghost particles usually change in different gauges but since we saw that they don't enter the R_2 part in any case we don't have to consider them [1] . In a general R_ξ gauge, the propagators of the scalar goldstone bosons and the vector bosons are modified as follows

$$S \;\overset{p}{\dashrightarrow}\; S \quad = \frac{i}{p^2 - \xi M_S^2}$$

$$V_\alpha \;\overset{p}{\rightsquigarrow}\; V_\beta \quad = \frac{-i}{p^2 - M_V^2}\left(g_{\alpha\beta} - (1-\xi)\frac{p_\alpha p_\beta}{p^2 - \xi M_V^2}\right) .$$

For the rest we use again the Feynman rules given in [28].Notice, that the $\xi = 1$ case reproduces the Feynman rules of the 't Hooft-Feynman gauge.

To compute our results in the Unitary gauge, we simply take the limit $\xi \to \infty$ in the above propagators *before integrating* over the loop momentum. Then the unphysical scalar particles decouple and the massive gauge boson propagators become

$$\frac{-i}{p^2 - M_V^2}\left(g_{\alpha\beta} - \frac{p_\alpha p_\beta}{M_V^2}\right) , \tag{5.1}$$

while for the photon we use

$$\frac{-i}{p^2} g_{\alpha\beta}. \tag{5.2}$$

Notice that the choice in Eq. 5.1 is mandatory in the framework of the OPP method. In fact, taking the limit $\xi \to \infty$ *after* integration over the loop momentum would imply a nonviable numerical cancellation between R_1 and R_2, since the two parts are treated separately. Our treatment for the cases that include γ_5 remains as before with a choice of a special vertex that we always start from.

[1]In the case of ghosts in the loop none of the extra integrals can appear. Same goes for unitary gauge and that is important because there, we would have to modify more the Feynman rules, to consider contributions from possible Yang-Lee terms

5.2 Results

We omit, in this chapter, the gauge invariant contributions coming from fermion loops, because they can be recovered with the help of the formulae we already worked out in the case of the 't Hooft-Feynman gauge in [22] In fact, the fermion loop part can be easily separated from the rest since it always involves a sum $\sum_i$ over fermions or fermion families. The same parameter λ_{HV} is again used in our formulae such that $\lambda_{HV} = 1$ corresponds to the 't Hooft-Veltman scheme and $\lambda_{HV} = 0$ to the FDH scheme of eq. 3.22.

We explicitly write down, all the formulae in the 2-point case, while, for the 3 the and 4-point vertices, we just classify the non vanishing ones due to the large size of the formulas we obtain. The results can be found as a FORM output in http://www.ugr.es/local/pittau/CutTools while in http://www.ugr.es/ garzelli/R2SM a package to compute the R_2 terms in different gauges is found. In [32] it is explained how to use this package to reproduce our results.

The notation used in those files closely follows that one introduced in the previous chapter. In Fig. 5.1-5.3 we present the generic non vanishing 2-point, 3-point and 4-point vertices that appear in our calculation, that also serve to further fix our notations.

5.2.1 Results in R_ξ gauge

Bosonic contribution to the vertices with 2 legs

Scalar-Scalar effective vertices

The generic effective vertex is

$$\mathrm{Vert}(S_1, S_2) = \frac{ie^2}{16\pi^2 s_w^2} C \tag{5.3}$$

with $\mathrm{Vert}(S_1, S_2)$ given in fig. 5.1 (a) and with the actual values of S_1, S_2 and C

$$HH \;\; : \;\; C \;\; = \;\; \frac{m_W^2}{4}\left(1 + 2\xi - \xi^2 - 12\lambda_{HV}\right)\left(1 + \frac{1}{2c_w^4}\right) + p_1^2\,\frac{9 - 11\xi}{24}\left(1 + \frac{1}{2c_w^2}\right)$$

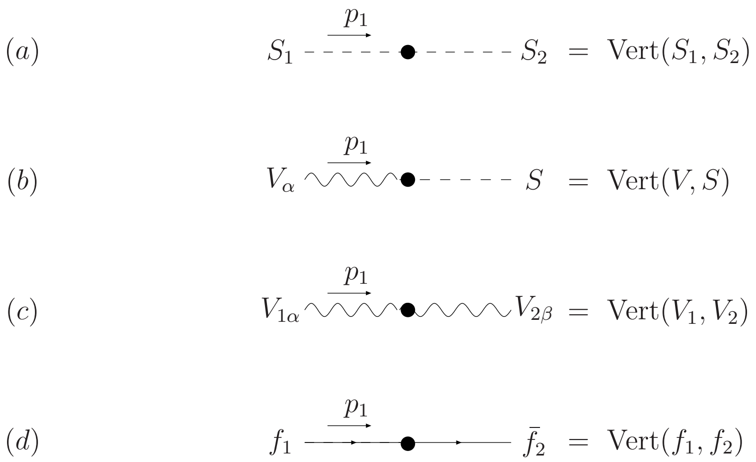


Figure 5.1: All possible 2-point vertices.

$$
\begin{aligned}
\chi\chi \quad : \quad C \quad &= \quad \frac{m_W^2}{24c_w^4}\left(1+2\xi^2-12\lambda_{HV}\right)+\frac{m_W^2}{12}\left(1-2\xi+7\xi^2-12\lambda_{HV}\right)\\
&\quad -\frac{m_H^2}{12c_w^2}\left(1-\frac{5}{2}\xi\right)+p_1^2\,\frac{9-11\xi}{24}\left(1+\frac{1}{2c_w^2}\right)\\
\phi^-\phi^+ \quad : \quad C \quad &= \quad \frac{m_W^2}{24c_w^4}\left(1+2\xi^2-12\lambda_{HV}\right)+\frac{m_W^2}{2c_w^2}\left(\xi-\frac{3}{2}\xi^2\right)\\
&\quad +\frac{m_W^2}{12}\left(1-8\xi+16\xi^2-12\lambda_{HV}\right)\\
&\quad -\frac{m_H^2}{12}\left(1-\frac{5}{2}\xi\right)+p_1^2\,\frac{9-11\xi}{24}\left(1+\frac{1}{2c_w^2}\right)
\end{aligned}
\tag{5.4}
$$

Vector-Scalar effective vertices

The generic effective vertex is

$$
\mathrm{Vert}(V,S)=\frac{ie^2}{\pi^2}\,C\,p_{1\alpha} \tag{5.5}
$$

with Vert(V, S) given in fig. 5.1 (b) and with the actual values of V, S and

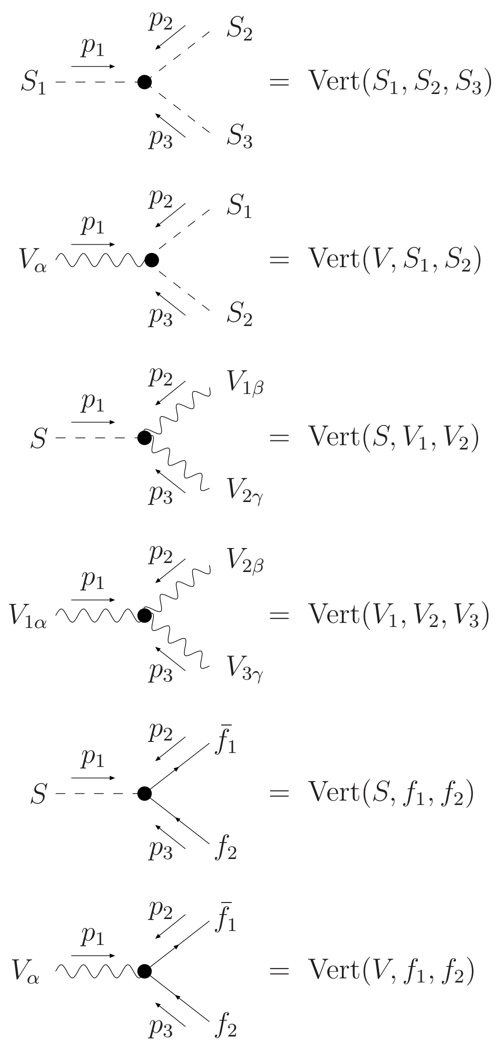
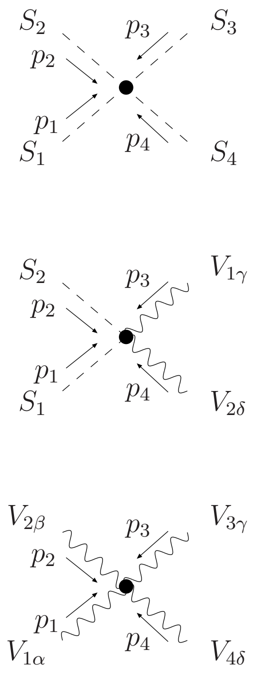


Figure 5.2: All possible nonvanishing 3-point vertices.



$$= \mathrm{Vert}(S_1, S_2, S_3, S_4)$$

$$= \mathrm{Vert}(S_1, S_2, V_1, V_2)$$

$$= \mathrm{Vert}(V_1, V_2, V_3, V_4)$$

Figure 5.3: All possible nonvanishing 4-point vertices.

C

$$
\begin{aligned}
W^-\phi^+ \quad &: \quad C \quad = \quad -(1-\xi)\frac{M_W}{192 c_w^2 s_w^2} \\
W^+\phi^- \quad &: \quad C \quad = \quad (1-\xi)\frac{M_W}{c_w^2 s_w^2} \\
Z\chi \quad &: \quad C \quad = \quad (1-\xi)\frac{i\,M_Z}{192 c_w^2 s_w^2}\left(1+2c_w^2 s_w^2\right) \\
A\chi \quad &: \quad C \quad = \quad (1-\xi)\frac{i\,M_Z}{96}\frac{c_w}{s_w}
\end{aligned}
\tag{5.6}
$$

Notice that all these vertices vanish in the 't Hooft-Feynman gauge ($\xi = 1$).

Vector-Vector effective vertices

The generic effective vertex is

$$
\mathrm{Vert}(V_1, V_2) = \frac{ie^2}{8\pi^2}\left(C_1\, p_{1\alpha} p_{1\beta} + C_2\, g_{\alpha\beta}\right) \tag{5.7}
$$

with $\mathrm{Vert}(V_1, V_2)$ given in fig. 5.1 (c) and with the actual values of V_1, V_2, C_1 and C_2

$$
\begin{aligned}
AA \quad : \quad C_1 \quad &= \quad K_1 \\
C_2 \quad &= \quad K_2 \\
\\
AZ \quad : \quad C_1 \quad &= \quad -\frac{c_w}{s_w}K_1 \\
C_2 \quad &= \quad -\frac{c_w}{s_w}K_2 \\
ZZ \quad : \quad C_1 \quad &= \quad \frac{c_w^2}{s_w^2}K_1 \\
C_2 \quad &= \quad \frac{c_w^2}{s_w^2}K_2 \\
W^-W^+ \quad : \quad C_1 \quad &= \quad \frac{1}{s_w^2}K_1 \\
C_2 \quad &= \quad \frac{1}{s_w^2}K_2
\end{aligned}
\tag{5.8}
$$

where

$$
\begin{aligned}
K_1 &= -\frac{1}{3}\lambda_{HV} + \frac{3}{4}(1-\xi) \\
K_2 &= p^2\left(\frac{21\xi - 17}{24} + \frac{\lambda_{HV}}{3}\right) - \xi\frac{\xi+3}{4}m_W^2
\end{aligned} \tag{5.9}
$$

Fermion-Fermion effective vertices

The generic effective vertex is

$$
\mathrm{Vert}(f_1, f_2) = \frac{ie^2}{\pi^2}\left[\left(C_-\Omega^- + C_+\Omega^+\right)\not{p}_1 + C_0\right] \tag{5.10}
$$

with $\mathrm{Vert}(f_1, f_2)$ given in fig. 5.1 (d) and with the actual values of f_1, f_2, C_-, C_+ and C_0

$$
\begin{aligned}
uu \;:\; C_- &= \frac{Q_u^2}{c_w^2}\left(\frac{\lambda_{HV}}{16} - \frac{1-\xi}{24}\right) \\
C_+ &= \left(\frac{I_{3u}^2}{s_w^2 c_w^2} - \frac{2Q_u I_{3u}}{c_w^2} + \frac{Q_u^2}{c_w^2} + \frac{1}{2s_w^2}\left(V_{ud}V_{du}^\dagger\right)\right)\left(\frac{\lambda_{HV}}{16} - \frac{1-\xi}{24}\right) \\
C_0 &= \frac{m_u Q_u}{8c_w^2}\left(Q_u - I_{3u}\right)\left(\lambda_{HV} - \frac{1-\xi}{4}\right) \\
dd \;:\; C_- &= \frac{Q_d^2}{c_w^2}\left(\frac{\lambda_{HV}}{16} - \frac{1-\xi}{24}\right) \\
C_+ &= \left(\frac{I_{3d}^2}{s_w^2 c_w^2} - \frac{2Q_d I_{3d}}{c_w^2} + \frac{Q_d^2}{c_w^2} + \frac{1}{2s_w^2}\left(V_{ud}V_{du}^\dagger\right)\right)\left(\frac{\lambda_{HV}}{16} - \frac{1-\xi}{24}\right) \\
C_0 &= \frac{m_d Q_d}{8c_w^2}\left(Q_d - I_{3d}\right)\left(\lambda_{HV} - \frac{1-\xi}{4}\right) \\
ll \;:\; C_- &= \frac{Q_l^2}{c_w^2}\left(\frac{\lambda_{HV}}{16} - \frac{1-\xi}{24}\right) \\
C_+ &= \left(\frac{I_{3l}^2}{s_w^2 c_w^2} - \frac{2Q_l I_{3l}}{c_w^2} + \frac{Q_l^2}{c_w^2} + \frac{1}{2s_w^2}\right)\left(\frac{\lambda_{HV}}{16} - \frac{1-\xi}{24}\right) \\
C_0 &= \frac{m_l Q_l}{8c_w^2}\left(Q_l - I_{3l}\right)\left(\lambda_{HV} - \frac{1-\xi}{4}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nu_l\nu_l \quad : \quad C_- &= 0 \\
C_+ &= \frac{1}{s_w^2}\left(\frac{I_{3\nu_l}^2}{c_w^2}+\frac{1}{2}\right)\left(\frac{\lambda_{HV}}{16}-\frac{1-\xi}{24}\right) \\
C_0 &= 0
\end{aligned}
\tag{5.11}
$$

Bosonic contribution to the vertices with 3 legs

The generic 3-point vertices appearing in our calculation are drawn in Fig. 5.2. As already pointed out, we limit ourselves to list the non vanishing cases. We found 43 non zero R_2 vertices in the R_ξ gauge, classified in Table 5.1.

Bosonic contribution to the vertices with 4 legs

All non vanishing generic 4-point vertices that appear in our calculation are drawn in Fig. 5.3. The 35 non zero R_2 vertices in the R_ξ gauge are classified in Table 5.2.

5.2.2 Results in the Unitary gauge

We follow again the notations of Fig. 5.1.

Bosonic contribution to the vertices with 2 legs

Scalar-Scalar effective vertices

The generic effective vertex is

$$
\mathrm{Vert}(S_1,S_2)=\frac{ie^2}{16\pi^2 s_w^2}C
\tag{5.12}
$$

with $\mathrm{Vert}(S_1,S_2)$ given in fig. 5.1 (a) and with the actual values of S_1, S_2 and C[2]

$$
HH \quad : \quad C \;=\; \frac{5}{6}p_1^2\left(1+\frac{1}{2c_w^2}\right)-\frac{9}{40}\frac{p_1^4}{m_W^2}-m_W^2\left(1+\frac{1}{2c_w^4}\right)\left(\frac{1}{4}+3\lambda_{HV}\right)
\tag{5.13}
$$

[2]Notice that in Unitary gauge all other scalar particles have decoupled from the theory

Scalar-Scalar-Scalar vertices:

$$\mathrm{Vert}(H,H,H),\quad \mathrm{Vert}(H,\chi,\chi),\quad \mathrm{Vert}(H,\phi^+,\phi^-).$$

Vector-Scalar-Scalar vertices:

$$\begin{array}{llll}
\mathrm{Vert}(A,H,\chi), & \mathrm{Vert}(A,\phi^+,\phi^-), & \mathrm{Vert}(Z,H,\chi), & \mathrm{Vert}(Z,\phi^+,\phi^-),\\
\mathrm{Vert}(W^-,H,\phi^+), & \mathrm{Vert}(W^-,\chi,\phi^+), & \mathrm{Vert}(W^+,H,\phi^-), & \mathrm{Vert}(W^+,\chi,\phi^-).
\end{array}$$

Scalar-Vector-Vector vertices:

$$\begin{array}{llll}
\mathrm{Vert}(H,A,A), & \mathrm{Vert}(H,A,Z), & \mathrm{Vert}(H,Z,Z), & \mathrm{Vert}(H,W^+,W^-),\\
\mathrm{Vert}(\phi^-,A,W^+), & \mathrm{Vert}(\phi^+,A,W^-) & \mathrm{Vert}(\phi^-,Z,W^+), & \mathrm{Vert}(\phi^+,Z,W^-).
\end{array}$$

Vector-Vector-Vector vertices:

$$\mathrm{Vert}(A,W^+,W^-),\quad \mathrm{Vert}(Z,W^+,W^-).$$

Scalar-Fermion-Fermion vertices:

$$\begin{array}{llll}
\mathrm{Vert}(H,u,u), & \mathrm{Vert}(H,d,d), & \mathrm{Vert}(H,l,l), & \\
\mathrm{Vert}(\chi,u,u), & \mathrm{Vert}(\chi,d,d), & \mathrm{Vert}(\chi,l,l), & \\
\mathrm{Vert}(\phi^-,d,u), & \mathrm{Vert}(\phi^-,l,\nu_l), & \mathrm{Vert}(\phi^+,u,d), & \mathrm{Vert}(\phi^+,\nu_l,l).
\end{array}$$

Vector-Fermion-Fermion vertices:

$$\begin{array}{llll}
\mathrm{Vert}(A,u,u), & \mathrm{Vert}(A,d,d), & \mathrm{Vert}(A,\nu_l,\nu_l), & \mathrm{Vert}(A,l,l),\\
\mathrm{Vert}(Z,u,u), & \mathrm{Vert}(Z,d,d), & \mathrm{Vert}(Z,\nu_l,\nu_l), & \mathrm{Vert}(Z,l,l),\\
\mathrm{Vert}(W^-,d,u), & \mathrm{Vert}(W^-,l,\nu_l), & \mathrm{Vert}(W^+,u,d), & \mathrm{Vert}(W^+,\nu_l,l).
\end{array}$$

Table 5.1: The 43 non zero 3-point effective vertices in the R_ξ gauge. In the Unitary gauge there are 23 non vanishing vertices, namely the 22 listed here that do not contain χ or $\phi^\pm$ fields plus $\mathrm{Vert}(H,\nu_l,\nu_l)$.

Scalar-Scalar-Scalar-Scalar vertices:

$$\begin{array}{lll}
\mathrm{Vert}(H,H,H,H), & \mathrm{Vert}(H,H,\chi,\chi), & \mathrm{Vert}(H,H,\phi^-,\phi^+), \\
\mathrm{Vert}(\chi,\chi,\chi,\chi), & \mathrm{Vert}(\chi,\chi,\phi^-,\phi^+), & \mathrm{Vert}(\phi^-,\phi^+,\phi^-,\phi^+).
\end{array}$$

Scalar-Scalar-Vector-Vector effective vertices:

$$\begin{array}{llll}
\mathrm{Vert}(H,H,A,A), & \mathrm{Vert}(H,H,A,Z), & \mathrm{Vert}(H,H,Z,Z), & \mathrm{Vert}(H,H,W^-,W^+), \\
\mathrm{Vert}(H,\phi^+,W^-,A), & \mathrm{Vert}(H,\phi^+,W^-,Z), & \mathrm{Vert}(\chi,\chi,A,A), & \mathrm{Vert}(\chi,\chi,A,Z), \\
\mathrm{Vert}(\chi,\chi,Z,Z), & \mathrm{Vert}(\chi,\chi,W^-,W^+), & \mathrm{Vert}(\chi,\phi^+,W^-,A), & \mathrm{Vert}(\chi,\phi^+,W^-,Z), \\
\mathrm{Vert}(\phi^-,H,A,W^+), & \mathrm{Vert}(\phi^-,H,Z,W^+), & \mathrm{Vert}(\phi^-,\chi,A,W^+), & \mathrm{Vert}(\phi^-,\chi,Z,W^+), \\
\mathrm{Vert}(\phi^-,\phi^+,A,A), & \mathrm{Vert}(\phi^-,\phi^+,A,Z), & \mathrm{Vert}(\phi^-,\phi^+,Z,Z), & \mathrm{Vert}(\phi^-,\phi^+,W^-,W^+).
\end{array}$$

Vector-Vector-Vector-Vector effective vertices:

$$\begin{array}{lll}
\mathrm{Vert}(A,A,A,A), & \mathrm{Vert}(A,A,A,Z), & \mathrm{Vert}(A,A,Z,Z), \\
\mathrm{Vert}(A,Z,Z,Z), & \mathrm{Vert}(Z,Z,Z,Z), & \mathrm{Vert}(A,A,W^-,W^+), \\
\mathrm{Vert}(A,Z,W^-,W^+), & \mathrm{Vert}(Z,Z,W^-,W^+), & \mathrm{Vert}(W^-,W^+,W^-,W^+).
\end{array}$$

Table 5.2: The 35 non zero 4-point effective vertices in the R_ξ gauge. In the Unitary gauge there are 14 non vanishing vertices, namely all those ones that do not contain χ or $\phi^\pm$ fields.

Vector-Scalar effective vertices

No contribution is found in the Unitary gauge.

Vector-Vector effective vertices

The generic effective vertex is

$$\mathrm{Vert}(V_1, V_2) = \frac{ie^2}{8\pi^2} \left(C_1\, p_{1\alpha} p_{1\beta} + C_2\, g_{\alpha\beta}\right) \tag{5.14}$$

with $\mathrm{Vert}(V_1, V_2)$ given in fig. 5.1 (c) and with the actual values of V_1, V_2, C_1 and C_2

$$\begin{aligned}
AA \quad : \quad C_1 &= K_1 \\
C_2 &= K_2 \\[2ex]
AZ \quad : \quad C_1 &= -\frac{c_w}{s_w} K_1 \\
C_2 &= -\frac{c_w}{s_w} K_2 \\[2ex]
ZZ \quad : \quad C_1 &= \frac{c_w^2}{s_w^2} K_1 \\
C_2 &= \frac{c_w^2}{s_w^2} K_2 \\
W^- W^+ \quad : \quad C_1 &= \frac{1}{s_w^2} K_3 \\
C_2 &= \frac{1}{s_w^2} K_4
\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}
K_1 &= -\frac{1}{3}\left(\lambda_{HV} - 5\right) - \frac{17}{60}\frac{p_1^2}{m_W^2} \\
K_2 &= \frac{3}{4} m_W^2 + \frac{1}{3} p_1^2 \left(\lambda_{HV} - \frac{23}{4}\right) + \frac{37}{120}\frac{p_1^4}{m_W^2} \\
K_3 &= -\frac{1}{3}\left(\lambda_{HV} - \frac{5}{2} - \frac{9}{8} c_w^2\right) + \frac{11}{24} c_w^4 - \frac{17}{120}\frac{p_1^2}{m_W^2}\left(1 + c_w^4\right)
\end{aligned}$$

$$\begin{aligned} K_4 &= \frac{3}{8}\frac{m_W^2}{c_w^2}\left(s_w^2+c_w^4+c_w^6\right)+p_1^2\left[\frac{\lambda_{HV}}{3}-\frac{7}{8}-\frac{7}{16}c_w^2\left(1+\frac{29}{21}c_w^2\right)\right] \\ &+\frac{37}{240}\frac{p_1^4}{m_W^2}\left(1+c_w^4\right) \end{aligned} \tag{5.16}$$

Fermion-Fermion effective vertices

The generic effective vertex is

$$\mathrm{Vert}(f_1,f_2)=\frac{ie^2}{\pi^2}\left[\left(C_-\Omega^-+C_+\Omega^+\right)\not{p}_1+C_0\right] \tag{5.17}$$

with $\mathrm{Vert}(f_1,f_2)$ given in fig. 5.1 (d) and with the actual values of f_1, f_2, C_-, C_+ and C_0

$$\begin{aligned} uu \; : \; C_- &= \frac{Q_u^2}{16c_w^2}\left[\lambda_{HV}+\frac{s_w^2}{m_Z^2}\left(\frac{p_1^2}{4}-\frac{2}{3}m_Z^2-\frac{5}{6}m_u^2\right)\right] \\ C_+ &= \frac{\lambda_{HV}}{16}\left[\frac{I_{3u}^2}{s_w^2c_w^2}-\frac{2Q_uI_{3u}}{c_w^2}+\frac{Q_u^2}{c_w^2}+\frac{1}{2s_w^2}\left(V_{ud}V_{du}^\dagger\right)\right] \\ &+\frac{s_w^2}{16m_Z^2c_w^2}\left(\frac{p_1^2}{4}-\frac{2}{3}m_Z^2-\frac{5}{6}m_u^2\right)\left(Q_u-\frac{I_{3u}}{s_w^2}\right)^2 \\ &+\frac{V_{ud}V_{du}^\dagger}{32m_W^2s_w^2}\left(\frac{p_1^2}{4}-\frac{2}{3}m_W^2-\frac{5}{6}m_d^2\right) \\ C_0 &= \frac{Q_um_u}{8c_w^2}\left[\lambda_{HV}\left(Q_u-I_{3u}\right)+\frac{s_W^2}{4m_Z^2}\left(Q_u-\frac{I_{3u}}{s_w^2}\right)\left(\frac{p_1^2}{3}-m_Z^2-m_u^2\right)\right] \\ dd \; : \; C_- &= \frac{Q_d^2}{16c_w^2}\left[\lambda_{HV}+\frac{s_w^2}{m_Z^2}\left(\frac{p_1^2}{4}-\frac{2}{3}m_Z^2-\frac{5}{6}m_d^2\right)\right] \\ C_+ &= \frac{\lambda_{HV}}{16}\left[\frac{I_{3d}^2}{s_w^2c_w^2}-\frac{2Q_dI_{3d}}{c_w^2}+\frac{Q_d^2}{c_w^2}+\frac{1}{2s_w^2}\left(V_{ud}V_{du}^\dagger\right)\right] \\ &+\frac{s_w^2}{16m_Z^2c_w^2}\left(\frac{p_1^2}{4}-\frac{2}{3}m_Z^2-\frac{5}{6}m_d^2\right)\left(Q_d-\frac{I_{3d}}{s_w^2}\right)^2 \\ &+\frac{V_{ud}V_{du}^\dagger}{32m_W^2s_w^2}\left(\frac{p_1^2}{4}-\frac{2}{3}m_W^2-\frac{5}{6}m_u^2\right) \\ C_0 &= \frac{Q_dm_d}{8c_w^2}\left[\lambda_{HV}\left(Q_d-I_{3d}\right)+\frac{s_W^2}{4m_Z^2}\left(Q_d-\frac{I_{3d}}{s_w^2}\right)\left(\frac{p_1^2}{3}-m_Z^2-m_d^2\right)\right] \end{aligned}$$

$$
\begin{aligned}
ll \quad : \quad C_- &= \frac{Q_l^2}{16c_w^2}\left[\lambda_{HV} + \frac{s_w^2}{m_Z^2}\left(\frac{p_1^2}{4} - \frac{2}{3}m_Z^2 - \frac{5}{6}m_l^2\right)\right] \\
C_+ &= \frac{\lambda_{HV}}{16}\left[\frac{I_{3l}^2}{s_w^2 c_w^2} - \frac{2Q_l I_{3l}}{c_w^2} + \frac{Q_l^2}{c_w^2} + \frac{1}{2s_w^2}\right] \\
&\quad + \frac{s_w^2}{16m_Z^2 c_w^2}\left(\frac{p_1^2}{4} - \frac{2}{3}m_Z^2 - \frac{5}{6}m_l^2\right)\left(Q_l - \frac{I_{3l}}{s_w^2}\right)^2 \\
&\quad + \frac{1}{32m_W^2 s_w^2}\left(\frac{p_1^2}{4} - \frac{2}{3}m_W^2\right) \\
C_0 &= \frac{Q_l m_l}{8c_w^2}\left[\lambda_{HV}\left(Q_l - I_{3l}\right) + \frac{s_W^2}{4m_Z^2}\left(Q_l - \frac{I_{3l}}{s_w^2}\right)\left(\frac{p_1^2}{3} - m_Z^2 - m_l^2\right)\right] \\
\nu_l\nu_l \quad : \quad C_- &= 0 \\
C_+ &= \frac{\lambda_{HV}}{16s_w^2}\left(\frac{1}{2} + \frac{I_{3\nu_l}^2}{c_w^2}\right) + \frac{I_{3\nu_l}^2}{16m_Z^2 c_w^2 s_w^2}\left(\frac{p_1^2}{4} - \frac{2}{3}m_Z^2\right) \\
&\quad + \frac{1}{32m_W^2 s_w^2}\left(\frac{p_1^2}{4} - \frac{2}{3}m_W^2 - \frac{5}{6}m_l^2\right) \\
C_0 &= 0
\end{aligned}
\tag{5.18}
$$

Bosonic contribution to the vertices with 3 legs

The generic 3-point vertices appearing in our calculation are drawn in Fig. 5.2. We found 23 non zero R_2 vertices in the Unitary gauge, classified in Table 5.1.

Bosonic contribution to the vertices with 4 legs

All non vanishing generic 4-point vertices that appear in our calculation are drawn in Fig. 5.3. The 14 non zero R_2 vertices in the Unitary gauge are classified in Table 5.2.

5.3 Checks

All our formulae have been obtained cross-checking two independent calculations. To further check our results, we used the fact that the $\mathrm{R} = \mathrm{R}_1 + \mathrm{R}_2$ contribution to physical quantities should be independent of the chosen gauge. In particular, parametrizing the gauge boson self-energies as follows

$$\Sigma_V^{\mu\nu}(p) \quad = \quad g^{\mu\nu}\, \Sigma_{V0}(p^2) + p^\mu p^\nu\, \Sigma_{V1}(p^2) \quad \text{with} \quad V = Z, W, \gamma\,, \tag{5.19}$$

we verified that the R contribution to $\Sigma_{W0}(M_W^2)$, $\Sigma_{Z0}(M_Z^2)$ and $\Sigma_{\gamma 0}(0)$ is the same in both the R_ξ and the Unitary gauge. In addition, in the case of both gauges, we checked all of the 2-point like Ward Identities presented in C involving $\mathrm{Vert}(S_1, S_2)$, $\mathrm{Vert}(V, S)$ and $\mathrm{Vert}(V_1, V_2)$.

To test the 3-point sector, we computed the $\mathrm{R} = \mathrm{R}_1 + \mathrm{R}_2$ contribution to the process $H \to \gamma\gamma$. Again, we found the same answer working in both gauges, obtaining an expression for R in full agreement with that one presented in [40]. As for the 4-point sector, we checked that, in the limit $\xi \to 1$, we fully reproduce the effective vertices presented in [22].

Finally, in the case of the R_ξ gauge, we computed R_2 using both the following two equivalent representations for the massive gauge boson propagators

$$\begin{aligned}
&-i\left(\frac{g_{\alpha\beta}}{p^2 - M_V^2} - (1-\xi)\frac{p_\alpha p_\beta}{(p^2 - M_V^2)(p^2 - \xi M_V^2)}\right) \quad \text{and} \\
&-i\left(\frac{g_{\alpha\beta}}{p^2 - M_V^2} - \frac{p_\alpha p_\beta}{M_V^2(p^2 - M_V^2)} + \frac{p_\alpha p_\beta}{M_V^2(p^2 - \xi M_V^2)}\right)\,,
\end{aligned} \tag{5.20}$$

always finding the same results. Since the two expressions lead to different integrals in the intermediate stages of the calculation, this provides a strong consistency check of our procedure.

As a last remark notice that, when working in the Unitary gauge, we take the limit $\xi \to \infty$ *before* integrating over the loop momentum. The fact that this gives the same result for R as in a generic R_ξ gauge in the above mentioned cases *provided the same prescription is used in the calculation of* R_1 is an explicit check of the equivalence of the limits $\xi \to \infty$ after or before the loop momentum integration in the definition of the Unitary gauge at 1-loop.

Chapter 6

Two-loop reductions at the integrand level

Since reduction methods at the integrand level proved to be very succesful at the one loop case, we would like now to extend them at two loops [48]. There are similar attempts in the literature (see i.e. [43], [42] and [47]).

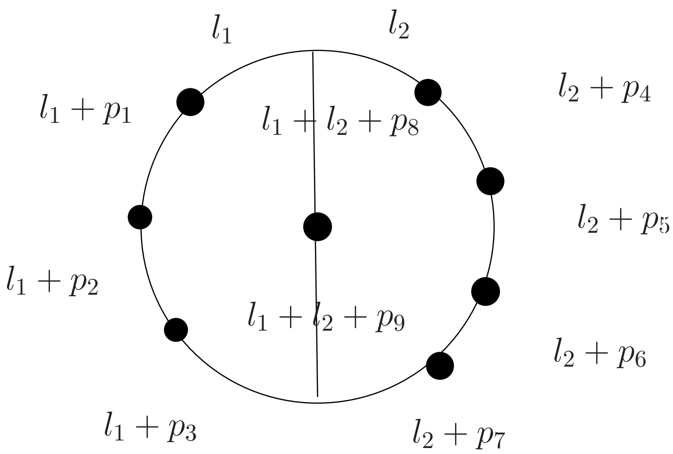
6.1 Definitions-Trivial decomposition

Let us assume that l_1 and l_2 are the two loop momenta. We consider three different kind of propagators for the three different loop lines of a generic two loop iGraph.

$$D(l_1 + p_i)\ ,\ D(l_2 + p_j)\ ,\ D(l_1 + l_2 + p_k) \tag{6.1}$$

where for instance $D(l_i + p_j) = (l_i + p_j)^2 - m_j^2$ and the p_j are the external momenta associated with the propagators of the diagram.

Such iGraphs can be denoted by the triple (n_1, n_2, n_3) which indicates the number of n_1 of propagators that contain only the one loop momentum l_1, the number n_3 of propagators containing only the other loop momentum l_2, and the n_2 propagators containing both. Obviously we have $(n_1, n_2, n_3) = (n_3, n_2, n_1)$, and also $(n_1, n_2, n_3) = (n_1, n_3, n_2)$ provided we also invert the n_1 external momenta going with l_1. Predictably, we write the order of the total iGraph as $n = n_1 + n_2 + n_3$.



An example of a two-loop iGraph, a (4,2,5) one.
One can see the three different loop lines.

The propagators depending on both loop momenta are called *mixed propagators*. If these are absent, the two integrals factor out and the problem becomes a double copy of one loop integrals. The same happens in case any other loop line is missing since, by shifting, one can always arrange the loop momenta such that they factor out. We consider these cases solved (by the one loop techniques) and will not discuss them further.

The requirement for trivial decomposition (with coefficients that are constants with respect to the loop momenta) now reads

$$\sum_{j=1}^{n_1} x_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} x_j D(l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} x_j D(l_1 + l_2 + p_j) = 1 \quad (6.2)$$

Proceeding in analogy with our one-loop discussion, we find that we have to satisfy the following equations :

$$\sum_{j=1}^{n_1+n_2} x_j = \sum_{j=n_1+1}^{n} x_j = \sum_{j=n_1+1}^{n_1+n_2} x_j = 0 \quad , \qquad (6.3)$$

$$\sum_{j=1}^{n_1+n_2} x_j {p_j}^{\mu} = \sum_{j=n_1+1}^{n} x_j {p_j}^{\mu} = 0 \quad , \qquad (6.4)$$

and

$$\sum_{j=1}^{n} x_j \mu_j = 1 \quad . \tag{6.5}$$

In total there are $2d + 4$ conditions, so that the minimum size of a trivially decomposable iGraph is $2d + 4$. In four dimensions, scalar iGraphs can therefore be decomposed down to $n = 11$. Again in analogy, for $n = 11$, since by shifting we can arrange $p_1 + \cdots + p_{n_1+n_2} = 0$ as well as $p_{n_1+1} + \cdots + p_n = 0$, the only solution to the $2d+3$ homogeneous equations is $x_j = 0$, $j = 1, \ldots, n$, and this fails the inhomogeneous equation. On the other hand, since any subset of an iGraph is itself an iGraph, any iGraph with $n_1 \geq 6$, $n_2 \geq 6$, or $n_3 \geq 6$ is trivially decomposable (for $d = 4$). Furthermore, with linear terms we see, from the one-loop discussion[1] that we only have to consider two-loop iGraphs with

$$n_{1,2,3} \leq 4 \quad , \quad n_1 + n_2 + n_3 \leq 11 \quad .$$

A word of caution is in order here. We may have a case where $n_1 + n_2 \geq 6$ and then decide to perform a decomposition *à la* one-loop with trivial terms, taking l_1 for the loop momentum. This is of course possible but note that in that case l_2 occurs as an external momentum. Since the solution of the linear equations is itself highly nonlinear due to matrix inversion, the resultant decomposition will not have the simple form of iGraphs again, and the resultant integrals will belong to very different classes of functions. It has been speculated that the simple iGraph form of integrands might be recovered if we combine *all* Feynman diagrams into our iGraph but since, for any given Feynman diagram at any given loop order, it is always possible [49] to find a theory where that particular Feynman diagram is, in fact, the *only* one, this does not seem likely at all, and we shall not pursue this further.

The number of two-loop iGraphs that we have to consider is therefore not very large : 4 for $d = 2$, 10 for $d = 3$, 19 for $d = 4$.

[1] In case there at least 6 propagators in one loop line we can first reduce the propagators in this loop line with constant coefficients and then continue further if possible. In the case of 5 propagators we already know that adding linear terms that depend only on the loop momentum of this loop line and take all coefficients that depend on the other loop momentum to zero, we can again solve the problem à la one loop.

6.2 Further reduction with linear terms

With trivial decomposition we see that we can always end up with an iGraph of order $2d+3$. Like in the one loop case, we now add coefficients linear in the loop momentum and hope for further reductions.

A note is in order here. In the one loop case the resulting integrals were always scalar. The reason is that any contraction of the loop momentum with any vector can either reconstruct denominators or be a spurious term. After integrating, in the case a denominator is reconstructed the remaining integral is a scalar integral with fewer denominators. In case the term is spurious it vanishes after integration. In two loops this is not the case anymore. One can always use dot products of the loop momenta with the momenta of the integrals to write relations like

$$2l_1 \cdot p_j = [(l_1 + p_j)^2 - m_j^2] - l_1^2 - p_j^2 + m_j^2 \quad . \tag{6.6}$$

The denominator $D(l_1 + p_j)$ may, however, not be present in the integral in case p_j appears in a propagator of another type such as $D(l_2 + p_j)^2$. Then, the product $l_1 \cdot p_j$ may be an *irreducible scalar product* (ISP) [42]. But not always; the ISP's of an integral are more complicated to write. For example, if there are enough propagators of the type $(l_1 + p_i)^2$ in the diagram such that the p_i's can form a base, one can rewrite p_j as a linear combination of the p_i's and manage to reconstruct denominators. There is a specific number of ISP's in any diagram and one can have some freedom in how to write them. The integrals of the resulting base can have numerators with ISP's in any power. It is not obvious that a scalar integral with a specific number of denominators is more difficult to calculate than a non-scalar integral with fewer denominators; however, from our experience we believe that this is the case.

Again, we want to write, if possible, the number 1 as in eq. (6.2). We use general linear terms in the sense that in every dimension we construct a base t_i (possibly, but not necessarily, the external momenta in the iGraph) and we have

$$x_j = \sum_i (a_j + b_{ij}(l_1 \cdot t_i) + c_{ij}(l_2 \cdot t_i)) \tag{6.7}$$

with the a, b, and c constants with respect to the loop momenta. Since in d dimensions, we need d vectors to construct such a base, it is obvious that for an iGraph of order n we start with $(2d+1)n$ coefficients. As in the one-loop case, we give a table that contains, for every dimension we worked with, the number of tensor structures it is possible to construct (denoted by $T(d)$) and the number of independent coefficients that we have explicitly calculated by cancellation probing as described above.

$$T(d) = (4d^2 + 18d + 2)/2$$

n	$d=6$	$d=5$	$d=4$	$d=3$	$d=2$	$d=1$
3	39-0	33-0	27-0	21-0	15-0	9-0
4	52-0	44-0	36-0	28-0	20-0	12-2
5	65-1	55-1	45-1	35-1	25-1	15-5
6	78-3	66-3	54-3	42-3	30-3	18-8
7	91-6	77-6	63-6	49-6	35-8	21-11
8	104-10	88-10	72-10	56-10	40-10	24-14
9	111-15	99-15	81-15	63-17	45-18	27-17
10	130-21	110-21	90-21	70-24	50-23	30-20
11	143-28	121-28	99-30	77-31	55-28	33-23
12	156-36	132-36	108-39	84-36	60-33	36-26
13	169-45	143-47	117-48	91-45	65-38	39-29
14	182-55	154-58	126-57	98-52	70-43	42-32
$T(d)$	127	96	69	46	27	10

In the table a line distinguishes between reducible and non-reducible cases. In all dimensions every $n = 2d + 2$ case is reducible with linear terms to a $n = 2d + 1$ iGraph. In four dimensions, we can decompose every integral down to integrals with 9 denominators. At this point, we are one step away from the limiting value of 8.

Since we mentioned unitarity, another remark is in order. We mentioned already that any iGraph with any $n_i > 4$ is reducible à la one loop and that is why we have a limited amount of iGraphs to reduce. It means that there might be iGraphs of order 9 or even less such as $(5, 1, 1)$ that is of order 7 but is still reducible with linear terms. This is in complete agreement with unitarity in the sense that these iGraphs don't have octuple cuts. In the example above we could try for an 7-fold cut but since 5 of the propagators depend only on one of the loop momenta, they cannot be all put to zero simultaneously[2].

[2] Unless we are talking about a very specific phase-space point where two propagators are equal; however, we do not discuss such cases.

6.2.1 Comments on the number of independent coefficients

The most difficult part in our counting is always the number of independent coefficients. As shown, we do it numerically but we would like to understand more the reason why we have as many independent coefficients as we do find. We try to demonstrate here a way to estimate this number; for the case of linear terms we will give some examples.

We can rewrite the terms in eq. (6.7). As we mentioned, terms of the form $l_i \cdot p_j$ either reconstruct denominators either become ISP. Let us assume the $(4, 1, 4)$ iGraph in four dimensions. It has in principle two ISP that we call σ_1 and σ_2. To see this, we note that in any of the loop lines that consists of four denominators, there are three external momenta. We can always "borrow" a fourth one from the other line to have a complete basis and write any product $l_i \cdot p_j$ as a linear combination of the four propagators and an ISP. The ISP in that case would be the product of l_i with the momentum we borrowed. Repeating for the other loop line we get the two ISP's. Using the coefficients of eq. (6.7) in eq. (6.2) we can either get a denominator times a constant or an ISP, or products of two denominators. We write this equation schematically as

$$1 = \sum_{i=1}^{9} D_i\{1, \sigma_1, \sigma_2\} + \sum_{i=1}^{9} \sum_{j,j\geq i}^{9} D_i D_j\{1\} \tag{6.8}$$

This means that by writing $\{1, \sigma_1, \sigma_2\}D_i$ there is a constant coefficient in front of every of these different terms:

$$\{1, \sigma_1, \sigma_2\} = a \cdot 1 + b_1\sigma_1 + b_2\sigma_2$$

where a, b_1, b_2 general numbers. In our particular example, we have $9 \times 3 + 45 \times 1 = 72$ coefficients. However, we started with a problem with up to cubic power in loop momenta and ended with up to quartic powers since we have these products of two denominators times some constants. These higher powers have to cancel, which means that we have to put extra constraints on our coefficients. We have 6 such constraints to cancel, namely the

$$l_1^4,\ l_1^2 l_2^2,\ l_2^4,\ (l_1 \cdot l_2) l_1^2,\ (l_1 \cdot l_2) l_2^2,\ (l_1 \cdot l_2)^2$$

terms. As a result we end up with $72 - 6 = 66$ independent coefficients, which is the number we get numerically as well. If we now try to decompose an iGraph of order 10 we can prove that eq. (6.8) becomes

$$1 = \sum_{i=1}^{10} D_i(1, \sigma_1, \sigma_2) + \sum_{i=1}^{9} \sum_{j,j \geq i}^{9} D_i D_j(1) \tag{6.9}$$

We don't need to go up to 10 in the product of 2 denominators since [3]

$$D_{10} \propto (D_1, ..., D_9, \sigma_1, \sigma_2)$$

We still need them, though, in the first term to produce terms of the type $(\sigma_i)^2$. In that case we have 69 independent coefficients and this graph is reducible. Adding more propagators we do not get more independent coefficients. In the same way one can count the independent coefficients in all dimensions although it is clear that it is safer to find their number numerically since there are a lot of overlaps in the tensor structures for higher cases. The way of rewriting the general linear terms as propagators and ISP's in the example above is still not the OPP method. In an extension of the OPP method to two loops, one would find the ISP's of every subdiagram and would avoid terms like D_i^2. We expect something similar to the one-loop case to happen then, rewriting σ_1 and σ_2 in the form of true ISP's of every subdiagram's contributions of the terms with the highest number of denominators to cancel. For that it is possible that special properties exist, as again in the one-loop case where spurious term solve a lot of equations by putting automatically tensor structures to zero.

6.2.2 Reduction with general quadratic terms

We try to reduce our iGraphs further with the use of general quadratic terms in the coefficients. In this case the coefficients become

$$\begin{aligned} x_j = \sum_i (&a_j + b_{ij}(l_1 \cdot t_i) + c_{ij}(l_2 \cdot t_i) + d_{ijk}(l_1 \cdot t_i)(l_1 \cdot t_k) \\ &+ e_{ijk}(l_2 \cdot t_i)(l_2 \cdot t_k) + f_{ijk}(l_1 \cdot t_i)(l_2 \cdot t_k)) \end{aligned} \tag{6.10}$$

We give the number of tensor structures $T(d)$ and the original number of coefficients with general quadratic terms, $C_1(d)$. The coefficients depend on

[3] This is actually the point where the number of dimensions plays a role in the counting.

the number of propagators n. Notice that the expression for $T(d)$ is not valid for $d = 2$. In that case, there is more overlap between the highest tensor structures. More specifically, for this particular case one can completely reconstruct the $l_1^2 l_2^\mu l_2^\nu$ structure from $l_2^2 l_1^\mu l_1^\nu$ and $(l_1 \cdot l_2) l_1^\mu l_2^\nu$ allowing fewer independent structures to be constructed.

$$
\begin{aligned}
T(d) &= 4d^3/3 + 10d^2 + 20d/3 - 2 && (6.11)\\
C_1(d) &= (2d^2 + 3d + 1)n && (6.12)
\end{aligned}
$$

With quadratic terms we start with $(2d^2+3d+1) \times n$ coefficients in total, not all of them of course being independent. Using cancellation probing we are able to find the number of independent coefficients for different iGraphs in different dimensions. We put our findings in the following tables :

quadratic terms, $d = 4$

$n = 3$		$n = 8$	
(1,1,1)	135-4	(1,3,4)	360-98
$n = 4$		(2,2,4)	360-98
(1,1,2)	180-6	(2,3,3)	360-98
$n = 5$		$n = 9$	
(1,1,3)	225-18	(1,4,4)	405-136
(1,2,2)	225-18	(2,3,4)	405-136
$n = 6$		(3,3,3)	405-136
(1,1,4)	270-38	$n = 10$	
(1,2,3)	270-38	(2,4,4)	450-180
(2,2,2)	270-38	(3,3,4)	450-180
$n = 7$		$n = 11$	
(1,2,4)	315-65	(3,4,4)	495-225
(1,3,3)	315-65	$n = 12$	
(2,2,3)	315-65	(4,4,4)	540-270

quadratic terms,$d = 3$

$n = 3$		$n = 6$	
(1,1,1)	84-3	(1,2,3)	168-32
$n = 4$		(2,2,2)	168-32
(1,1,2)	128-6	$n = 7$	
$n = 5$		(1,3,3)	196-53
(1,1,3)	140-16	(2,2,3)	196-53
(1,2,2)	140-16	$n = 8$	
		(2,3,3)	224-80
		$n = 9$	
		(3,3,3)	252-108

quadratic and cubic terms, $d = 2$

n	iGraph	quadratic	cubic
3	(1,1,1)	45-3	105-15
4	(1,1,2)	60-6	140-32
5	(1,1,3)	75-17	
	(1,2,2)	75-15	175-61
6	(2,2,2)	90-30	210-96

At the same time we can calculate the number of possible tensor structures in different dimensions. We give the results for some cases below:

d	2	3	4	5
Tensor Structures	60	144	270	448

Comparing the number of independent coefficients with the number of possible tensor structures we see that we can decompose in 2 dimensions an iGraph of order 5 to lower order iGraphs as indicated by unitarity. In this specific case we can go to an iGraph of order $2d = 4$. Indeed, we solve numerically this system and we get valid solutions. In the way we solve the problem we can always take values of the loop momenta and construct as many equations as unknowns and get a solution. A solution is valid only if at the end, the $1 = 1$ test is satisfied for any value of the loop momenta. We can say that we solved our task in the case of 2 dimensions. However, there is no solution in 3 and 4 dimensions and in this case we have to investigate what happens if we add cubic, quartic terms and so on. The two-dimensional case is exceptional here because of the extra properties that lower the number of tensor structures.

6.2.3 Reduction with general cubic terms

We focus now on 3 and 4 dimensions since we finished the reduction in 2 dimensions. We include now general cubic terms and our coefficients become

$$x_j = \sum_i (a_j + b_{ij}(l_1 \cdot t) + c_{ij}(l_2 \cdot t) + d_{ijk}(l_1 \cdot t)^2 + e_{ijk}(l_2 \cdot t)^2$$
$$+f_{ijk}(l_1 \cdot t)(l_2 \cdot t) + g_{ijkl}(l_1 \cdot t)^3$$
$$+h_{ijkl}(l_1 \cdot t)^2(l_2 \cdot t) + i_{ijkl}(l_2 \cdot t)^2(l_1 \cdot t) + j_{ijkl}(l_2 \cdot t)^3) \tag{6.13}$$

We give the number of tensor structures $T(d)$ and the original number of coefficients with general cubic terms $C_1(d)$. The coefficients depend on the number of propagators n. The expression for T(d) is valid for $d \geq 3$

$$T(d) = 2d^4/3 + 22d^3/3 + 71d^2/6 + d/6 + 1 \tag{6.14}$$
$$C_1(d) = (4d^3/3 + 4d^2 + 11d/3 + 1)n \tag{6.15}$$

That means that in d dimensions we start with $C_1(d)$ coefficients, not all of them being independent. We run the MAPLE code in the case of the iGraph of order 7 in 3 dimensions and we find that out of 588 original coefficients, 360 are independent. This is the number of tensor structures as well. Using another PYTHON-based program, we can actually solve the system decomposing any iGraph of order 7 in 3 dimensions to lower iGraphs with general cubic terms, and perform the 1=1 test. This means that with cubic terms we are able to decompose any two-loop iGraph in 3 dimensions to up to a $2d$ iGraph as expected from unitarity. In the same way, we can investigate $d = 4$, and we get a valid decomposition: from our original 1485 coefficients, 831 are independent and all the tensor structures can be reconstructed. Actually, we did the same in 5 dimensions as well and again we managed to decompose every integral up to integrals of order 10 using general cubic terms. We believe that this is a general result for any dimension, except of course for $d = 2$.

Summarising, at the two-loop level, ultimately cubic terms are needed (quadratic for $d = 2$)for a decomposition of any integral to integrals with up to $2d$ denominators. The decomposition is seen to lead to non-scalar, non-vanishing integrals. If one wants to design a two-loop OPP method, it is clear that one has to take our general linear,quadratic,cubic terms and rewrite them in terms of propagators (this would lead to contributions with less denominators) and ISP's of each subdiagram seperately [4]. Our work is basically the starting point of an OPP method and a proof that reductions that were conjectured in [43], [42], [47] are actually valid and survive the global $1-1$ test, in case one includes all relevant cuts.

[4] In the same way that one-loop OPP has different spurious terms for every subdiagram

Chapter 7

Integration By Parts at the integrand level

7.1 Introduction

In [45], a very useful technique for the study of multiloop calculations was introduced. It goes by the name Integration By Parts (IBP) and it uses the fact that any loop integral of a total derivative in dimensional regularisation vanishes. In other words, in dimensional regularisation, there are no contributions from surface terms. In order to understand better what does this mean, one could look at the simple example in [45], where they consider a loop integral in coordinate space. The IBP of this integral is then trivially zero since, the numerator of the integral vanishes identically. One can understand better the role of IBP's in integral reduction by checking [46] where a lot of examples are given.

Just to explain a little bit more let us assume a loop integral,i.e.:

$$I = \int \int d^d l_1 d^d l_2 \frac{1}{D(l_1 + p_1) D(l_2 + p_2) D(l_1 + l_2 + p_3)} \tag{7.1}$$

, where for simplicity we consider all denominators massless. Then, a typical IBP one would write is

$$\int \int d^d l_1 d^d l_2 \frac{\partial}{\partial l_1^\mu} \left(\frac{(l_1 + p_1)^\mu}{D(l_1 + p_1) D(l_2 + p_2) D(l_1 + l_2 + p_3)} \right) = 0 \tag{7.2}$$

Evaluating the IBP we get

$$
\begin{aligned}
& dI - \int \frac{2(l_1+p_1)^2}{D(l_1+p_1)^2 D(l_2+p_2) D(l_1+l_2+p_3)} - \\
& \int \frac{2(l_1+l_2+p_3)\cdot(l_1+p_1)}{D(l_1+p_1) D(l_2+p_2) D(l_1+l_2+p_3)^2} = 0
\end{aligned}
\tag{7.3}
$$

or in an even more simple way

$$
\begin{aligned}
& (d-3)I - \int \frac{1}{D(l_2+p_2) D(l_1+l_2+p_3)^2} + \\
& \int \frac{(l_2+p_3-p_1)^2}{D(l_1+p_1) D(l_2+p_2) D(l_1+l_2+p_3)^2} = 0
\end{aligned}
\tag{7.4}
$$

In principle, one should write all the IBP's coming from derivatives with respect to all loop momenta and for every combination of momenta in the numerator. Then, one gets a number of relations between Feynman integrals and by solving them, one can reduce the original integral. Obviously, the integral in general does not have to be scalar, we just take a scalar one for simplicity.

Notice that since there are derivatives included, IBP's can also yield relations between integrals with normal propagators and integrals with propagators with a higher power.

We want now to include IBP identities at the integrand level [48] . Since all the total derivatives of an integral vanish within the framework of dimensional regularisation, in all the reductions we performed so far at the integrand level, we can always add extra coefficients with total derivatives. In a sense, they are a type of spurious terms. They differ with the spurious terms we had before in the sense that spurious terms vanish not because the integral vanishes, but because the output of the integral is contracted with something orthogonal to it. Here, these terms vanish because of the integration. We will provide some examples where use of IBP's are made at the integrand level. With the addition of the IBP's we can extend OPP method in case i.e. of doubled propagators.

7.2 Examples of IBP's at the integrand level

We consider as a first example the following integral

$$\int \frac{1}{D_0^2 D_1} = \int \frac{1}{(q^2)^2 (q+p_1)^2} \tag{7.5}$$

It's a one loop example with a doubled propagator and massless particles. We want to perform a reduction of the type

$$\frac{1}{D_0^2 D_1} \to \frac{1}{D_0 D_1} \tag{7.6}$$

We make use of the following IBP identities

$$\begin{aligned} &\int \frac{\partial}{\partial q^\mu} \frac{(q+p_1)^\mu}{D_0 D_1} = 0 \\ &\int \frac{\partial}{\partial q^\mu} \frac{q^\mu}{D_0^2} = 0 \end{aligned} \tag{7.7}$$

At the integrand level we require

$$\frac{1}{D_0^2 D_1} = a_1 \frac{1}{D_0 D_1} + a_2 \frac{\partial}{\partial q^\mu} [\frac{(q+p_1)^\mu}{D_0 D_1}] + a_3 \frac{\partial}{\partial q^\mu} [\frac{q^\mu}{D_0^2}] \tag{7.8}$$

for some coefficients a_1, a_2, a_3. Multiplying with $D_0^2 D_1$ we get

$$1 = a_1 D_0 + a_2[(d-3)D_0 - D_1 + p_1^2] + \tilde{a}_3 D_1 \tag{7.9}$$

with $\tilde{a}_3 = a_3(d-4)$.eq. (7.9) is solved for $a_2 = \tilde{a}_3$, to cancel the D_1 term, $a_1 = (3-d)a_2$, to cancel the D_0 term and $a_2 = \frac{1}{p_1^2}$ so the whole thing is equal to one. Integrating, and since the second and third term vanish upon integration, we get

$$\int \frac{1}{D_0^2 D_1} = \frac{3-d}{p_1^2} \int \frac{1}{D_0 D_1} \tag{7.10}$$

which is the correct answer. We take now a second example to reduce

$$\frac{1}{D_0^2 D_1^2} \rightarrow \frac{1}{D_0 D_1} \tag{7.11}$$

At the integrand level this can be written (using again IBP indentities) as

$$\begin{aligned}1 = a_1 D_0 D_1 + a_2 D_0 + a_3 D_1 + a_4 D_0 D_1 (d - 3 + \frac{p_1^2 - D_1}{D_0}) + \\ a_5 D_0 (d - 5 + \frac{p_1^2 - D_1}{D_0}) + a_6 D_1^2\end{aligned} \tag{7.12}$$

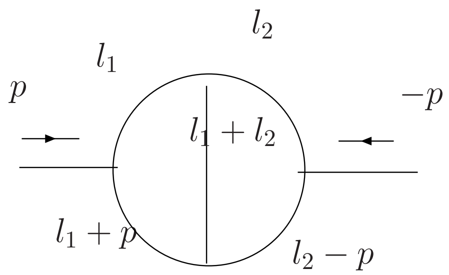
One can now solve this system as before and get the coefficients. However, to arrive at the final result one should use also either eq. (7.10) or eq. (7.9) to find

$$\int \frac{1}{D_0^2 D_1^2} = \frac{1}{(p_1^2)^2}((d-6)(d-3)) \int \frac{1}{D_0 D_1} \tag{7.13}$$

Notice that it is important to perform this reduction to include all the IBP's since the integrand could always lead to a middle step first before the final reduction and there lower IBP's are needed (lower in the sense that might include lower number of propagators).

We would like finally to present a two loop example with the use of IBP's at the integrand level. We take a (2,1,2) Feynman diagram in two dimensions with massless particles. The integral we have to consider is the following:

$$\int \frac{1}{l_1^2 (l_1 + p)^2 (l_1 + l_2)^2 l_2^2 (l_2 - p)^2} \tag{7.14}$$



A (2,1,2) two-loop Feynman diagram

This example can also be found in [45] and [46]. For this specific diagram, we did not manage to find a valid solution numerically with the previous methods and although the general (2,1,2) iGraph is decomposable this one seems not to be.The reason is that this case is a very restricted one since all propagators are massless and the whole diagram depends on one external momentum only. However, with the use of IBP's we can show, not only that it is decomposable, but that the result of its decomposition is even simpler diagrams than we would expect. We want to write number 1 (the numerator of the integral) in terms of denominators and terms that vanish upon integration due to IBP identities. We use all such possible terms. Then we ask for a solution numerically. We find a valid solution (valid again in the sense that at the end we perform a 1=1 test for all values of l_1 and l_2 and this is always satisfied). At the end we can integrate our result to get rid of the terms that vanish upon integration and we get

$$\int \frac{1}{l_1^2(l_1+p)^2(l_1+l_2)^2 l_2^2(l_2-p)^2} = \frac{-1}{p^2}\int \frac{1}{l_1^2(l_1+p)^2 l_2^2(l_2-p)^2} + \frac{4}{(p^2)^2}\int \frac{1}{l_1^2(l_1+l_2)^2(l_2-p)^2} \tag{7.15}$$

which once again agrees with what is known in the literature.

We showed explicitely some examples of reducing integrals by using IBP identities at the integrand level. Reduction methods at the integrand level, such as the OPP method, proved to be very powerful for the calculation of one loop amplitudes. We believe that in the two-loop case, although there is still a lot of work to be done, reductions at the integrand level can also play an important role. We saw how one by counting coefficients and tensor structures can perform such reductions and take results that are expected from unitarity. Integration by parts identities are also important, especially when considering higher loops [1]. We showed how one can also implement them and use them at the integrand level. In practice, a combination of these two things is what should be used to complete a reduction scheme of two-loop amplitudes.

[1] Of course there is nothing wrong with using them at one loop as well

Chapter 8

Conclusions

We arrive now at the conclusions of the thesis. In [15], a new method for reducing any one loop amplitude in terms of a set of scalar Feynman integrals was presented. The method works at the Integrand level and is suitable for a numerical implementation in a computer program (i.e. [17]).

The OPP method has been used in numerous one loop calculations so far and this proves its usefulness. In this thesis, we started by understanding all the features of this method and generally of all possible one loop reductions at the Integrand level. We understood why and in which cases one can achieve such reductions, starting in a completely general way, by using simple linear algebra. We wrote polynomial equations and investigated in which cases they can be solved. Since we dealt with scalar integrals mainly, we basically found fancy ways to write the number one in terms of denominators times coefficients:

$$D_1 T_1(q) + D_2 T_2(q) + \cdots + D_n T_n(q) = 1 \quad , \tag{8.1}$$

The equation above is not always solved. One has to count tensor structures for different dependence of $T_i(q)$ in the loop momentum and compare them with the number of independent coefficients. We found that for d integer dimensions, it is always possible to achieve reduction up to d denominators with coeffients (up to) linear in the loop momentum. Since our reduction is different with the OPP reduction we compared the results in some examples. We proved that the reductions are the same and the result is thus unique.

When departing from four dimensions (when assuming dimensional regularization) the OPP method has to be refined accordingly to the scheme that is followed. In our cases we used schemes when only the loop momentum is living in $4+\epsilon$ dimensions while the external momenta are kept strictly four dimensional. Splitting the numerator of any one loop amplitude in a four and an ϵ dimensional piece, a rational part arises together with the cut-constructible part that is described in the pure four dimensional OPP method. The mismatch in dimensions of the four dimensional piece of the numerator and the d dimensional denominators leads to the so-called R_1 part, while the part coming from the ϵ dimensional piece of the numerator is called R_2. In order to calculate the R_2 part, extra Feynman rules are needed. These Feynman rules can be extracted from the R_2 contribution of all one particle irreducible diagrams up to four denominators. Given a theory, one can calculate these Feynman rules and then perform a tree-level calculation (like when having counterterms) to obtain the R_2 piece of a process. Since we focus in Standard Model calculations (the OPP method can be used for any theory) there was a need to calculate these R_2 counterterms and complete all the necessary tools for a one loop calculation. We provided these Feynman rules for the biggest part of the Standard Model, the electroweak part [1]. The number of Feynman diagrams we dealt with was of the order of 10.000. We performed this calculation in different gauges such as the 't Hooft-Feynman gauge, a general R_ξ gauge and the Unitary gauge. We studied gauge dependence of the two rational pieces and we found that seperately R_1 and R_2 are not gauge invariant. However, when considered together, the whole rational part is.

At the last part of the thesis we repeated the "counting to one" part for two loop cases. We coould prove that any two loop integral can be written in terms of integrals with up to 2d denominators, in d dimensions. This is an important result since we would like at some moment to extend unitarity based methods, such as the OPP, in two loops. In this case, the coefficients we used must have up to cubic dependence in the loop momenta. In two dimensions a quadratic dependence is sufficient due to some properties of the tensor structures. We proved the need for cubic terms in three, four and five dimensions and we strongly believe that it holds for any other integer dimension.

Looking at results in the literature for specific cases, we found that Feynman integrals could be in some cases further reduced. We also had a counterexample in two dimensions, where although it is reduced as we expected, it

[1]The QCD and the QED part can be found in references [21] and [18]

was not possible to get this reduction with a unitarity based method. When one applies the maximal cut, all propagators could be set to zero which contradicts our $1 = 1$ test. In both cases we manage to solve the problem by adding extra terms in our master formula, terms that being total derivatives vanish upon integration, as described in the integration-by-parts method [45]. Combining the OPP method with integration-by-parts identities at the integrand level, one is able to reduce any Feynman integral into a minimal set of smaller Feynman integrals that are called Master Integrals. As a by product, the OPP method at one loop can be generalised for propagators of arbitrary power.

There is still a lot of work to be done to complete the two loop scheme. We believe that our counting method is a sufficient proof that such a unitarity based method can exist at two loops and in case one wants to achieve reduction to true master integrals a combination of OPP and IBP should be considered.

Appendix A

Triangles to Bubbles in two dimensions

In this appendix we discuss the decomposition of scalar triangles in two dimensions. The reason to move to two dimensions is that things there are simpler and one can get a better understanding of similar things in higher dimensions. We try to decompose a triangle with spurious terms and with general linear terms and we compare the results.

We want to decompose the triangle with the integrand

$$\frac{1}{D_0 D_1 D_2}$$

For simplicity and without loss of generality we took $p_0 = 0$. In principle any basis in two dimensions would do but we choose a specific basis consisting of the two following vectors

$$t_1^\mu = \epsilon^\mu(p_1), \quad t_2^\mu = \epsilon^\mu(p_2) \tag{A.1}$$

where

$$\epsilon^\mu(p_i) = \epsilon^{\mu\nu} p_{i\nu}$$

and $\epsilon^{\mu\nu}$ is the Levi-Civita tensor in two dimensions.

We now write eq. (1.3) for the numerator of the triangle

$$1 = \sum_{i=0}^{2} [a_i + \sum_{j=1}^{2} b_{ij}(q \cdot t_j)] D_i \tag{A.2}$$

We chose the particular basis because it is easy to recognise the spurious terms hidden in the general linear terms. We have the following spurious terms for this case

$$\begin{aligned} S_0 &= q_\mu(\epsilon^\mu(p_2) - \epsilon^\mu(p_1)) + \epsilon(p_1, p_2) \\ S_1 &= q_\mu \epsilon^\mu(p_2) \\ S_2 &= q_\mu \epsilon^\mu(p_1) \end{aligned} \tag{A.3}$$

Then eq. (A.2) can be written as

$$\begin{aligned} &1 = [a_0 + b_{02} S_0] D_0 + [a_1 + b_{12} S_1] D_1 + [a_2 + b_{21} S_2] D_2 + \\ &(b_{01} + b_{02}) q_\mu \epsilon^\mu(p_1) D_0 - b_{02} \epsilon(p_1, p_2) D_0 + \\ &b_{11} q_\mu \epsilon^\mu(p_1) D_1 + b_{22} q_\mu \epsilon^\mu(p_2) D_2 \end{aligned} \tag{A.4}$$

We decompose the same integrand now with the OPP method, using only spurious terms

$$1 = [a'_0 + b'_{02} S_0] D_0 + [a'_1 + b'_{12} S_1] D_1 + [a'_2 + b'_{21} S_2] D_2 \tag{A.5}$$

We see trivially that with the choice

$$\begin{aligned} b_{01} + b_{02} &= 0 \\ b_{11} = b_{22} &= 0 \end{aligned} \tag{A.6}$$

the two reductions can give identical solutions.We have

$$\begin{aligned} &a_0 - b_{02} \epsilon(p_1, p_2) = a'_0, \quad b_{02} = b'_{02}, \quad a_1 = a'_1 \\ &b_{12} = b'_{12}, \quad a_2 = a'_2, \quad b_{21} = b'_{21} \end{aligned} \tag{A.7}$$

However, when solving the system in eq. (A.4) we still have some freedom since we start with 9 coefficients to construct 8 tensor structures. That means that we could in principle find different solutions than the one above [1]. We explicitely checked this with MAPLE. Notice that from bubbles with a numerator linear in q (that exist in our first reduction) one can decompose to scalar tadpoles but scalar bubbles as well à la Passarino-Veltman

$$
\begin{aligned}
&\int \frac{q^\mu}{D_0 D_i} = \frac{p_i^\mu}{2p_i^2}\left[\int \frac{1}{D_0} - \int \frac{1}{D_i} + Y_{i0}\int \frac{1}{D_0 D_i}\right] \\
&\int \frac{q^\mu}{D_1 D_2} = \frac{(p_2-p_1)^\mu}{2(p_2-p_1)^2}\int \frac{1}{D_0} - \left(\frac{(p_2-p_1)^\mu}{2(p_2-p_1)^2} + p_1^\mu\right)\int \frac{1}{D_2} + \\
&\frac{(p_2-p_1)^\mu}{2(p_2-p_1)^2} Y_{2-1,0}\int \frac{1}{D_1 D_2}
\end{aligned}
\tag{A.8}
$$

with $Y_{i0} = m_i^2 - m_0^2 - p_i^2$ and $Y_{2-1,0} = m_2^2 - m_1^2 - (p_2-p_1)^2$.

What happens is that once you do this decomposition the scalar bubbles that enter, add to the rest of the bubbles and give the exact same decomposition à la OPP, while the tadpoles always vanish. This proves the uniqueness of the reduction.

[1]The solution above is special as we described in the case of pentagons with spurious terms in 4 dimensions. By choosing eq. (A.6) we are actually left with 6 coefficients from 9, however, we find a solution since the $q^\mu q^\nu$ part vanishes except for its trace solving 3-1=2 equations in one go.

Appendix B

List of extra integrals for the Rational part

We present the extra integrals that are needed for the calculation of the rational part. For the R_1 part integrals with powers of $\tilde{q}^2$ in the numerator are needed while for the R_2 part also Pole Parts (P.P.) of ultraviolet divergent integrals are necessary in schemes like 't Hooft-Veltman where the ϵ dependence of the numerator is kept.

B.1 Two-point integrals

$$
\begin{aligned}
\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j} &= -\frac{i\pi^2}{2}\left[m_i^2+m_j^2-\frac{(p_i-p_j)^2}{3}\right]+O(\epsilon)\,,\\
P.P.\left(\int d^n\bar{q}\frac{1}{\bar{D}_i\bar{D}_j}\right) &= -2\frac{i\pi^2}{\epsilon}\,,\\
P.P.\left(\int d^n\bar{q}\frac{q_\mu}{\bar{D}_i\bar{D}_j}\right) &= \frac{i\pi^2}{\epsilon}(p_i+p_j)_\mu\,,\\
P.P.\left(\int d^n\bar{q}\frac{q_\mu q_\nu}{\bar{D}_i\bar{D}_j}\right) &= \frac{i\pi^2}{3\epsilon}\Big\{\frac{(p_i-p_j)^2-3m_i^2-3m_j^2}{2}g_{\mu\nu}\\
&\quad -2\,p_{i\mu}p_{i\nu}-2\,p_{j\mu}p_{j\nu}\\
&\quad -p_{i\mu}p_{j\nu}-p_{j\mu}p_{i\nu}\Big\}\,. \qquad \text{(B.1)}
\end{aligned}
$$

B.2 Three-point integrals

$$
\begin{aligned}
\int d^n\bar{q}\frac{\tilde{q}^2}{\bar{D}_i\bar{D}_j\bar{D}_k} &= -\frac{i\pi^2}{2} + O(\epsilon)\,, \\
\int d^n\bar{q}\frac{\tilde{q}^2 q_\mu}{\bar{D}_i\bar{D}_j\bar{D}_k} &= \frac{i\pi^2}{6}(p_{ijk})_\mu + O(\epsilon)\,, \\
P.P.\left(\int d^n\bar{q}\frac{q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k}\right) &= -\frac{i\pi^2}{2\epsilon} g_{\mu\nu}\,, \\
P.P.\left(\int d^n\bar{q}\frac{q_\mu q_\nu q_\rho}{\bar{D}_i\bar{D}_j\bar{D}_k}\right) &= \frac{i\pi^2}{6\epsilon}\left[g_{\mu\nu}(p_{ijk})_\rho + g_{\nu\rho}(p_{ijk})_\mu + g_{\mu\rho}(p_{ijk})_\nu\right]
\end{aligned} \tag{B.2}
$$

B.3 Four-point integrals

$$
\begin{aligned}
\int d^n\bar{q}\frac{\tilde{q}^4}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} &= -\frac{i\pi^2}{6} + O(\epsilon)\,, \\
\int d^n\bar{q}\frac{\tilde{q}^2 q_\mu q_\nu}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} &= -\frac{i\pi^2}{12} g_{\mu\nu} + O(\epsilon)\,, \\
\int d^n\bar{q}\frac{\tilde{q}^2 q^2}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l} &= -\frac{i\pi^2}{3} + O(\epsilon)\,, \\
P.P.\left(\int d^n\bar{q}\frac{q_\mu q_\nu q_\rho q_\sigma}{\bar{D}_i\bar{D}_j\bar{D}_k\bar{D}_l}\right) &= -\frac{i\pi^2}{12\epsilon}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right).
\end{aligned} \tag{B.3}
$$

with $p_{ijk} = p_i + p_j + p_k$.

Appendix C

Ward identities with the BFM

The Background Field Method (BFM) is a technique for quantizing gauge theories without losing explicit gauge invariance of the effective action [34, 36, 37, 38, 39]. Starting from a classical Lagrangian, one can achieve this by decomposing the usual fields into background fields and quantum fields. Then, the background fields are treated as external sources, while the quantum fields are variables of integration in the functional integral. A gauge fixing term is added, which only breaks the invariance with respect to the quantum gauge transformations, while the invariance with respect to background-field gauge transformations is preserved. From the Lagrangian mentioned above, one can construct an effective action $\Gamma[\hat{V}, \hat{S}, F, \bar{F}]$, where $\hat{V}$ refers to the background gauge fields, $\hat{S}$ to the background scalar fields and $F, \bar{F}$ to the fermion fields (for all fields that do not enter the gauge-fixing term, quantization is identical in the BFM and in the conventional formalism. Their Feynman rules for the background fields and quantum fields are also identical, so there is no need to distinguish them). This effective action is invariant under the background gauge transformations given in eqs. 21, 22 of [34]. This invariance implies that

$$\frac{\delta\Gamma}{\delta\hat{\theta}^a} = 0\,, \tag{C.1}$$

where $a = A, Z, W^\pm$ and $\hat{\theta}^a$ are the infinitesimal gauge transformations of the background fields. By combining these formulas with eqs. 21, 22 of [34], one can produce eqs. 4, 5 and 6 of [38]. By differentiating them with respect to background fields and setting the fields equal to zero, one obtains Ward identities for the vertex functions that are precisely the Ward identities related to the classical Lagrangian. In the papers [34] and [38] some of these Ward identities are listed (see also [40]). In the following, we extend this list by producing more Ward identities useful for our checks [1].

C.1 Ward identities involving VV, VS and SS

$$k^\mu \Gamma^{AA}_{\mu\nu}(k,-k) = k^\mu \Gamma^{AZ}_{\mu\nu}(k,-k) = 0 \tag{C.2}$$

$$k^\mu \Gamma^{ZZ}_{\mu\nu}(k,-k) - iM_Z \Gamma^{\chi Z}_{\nu}(k,-k) = 0 \tag{C.3}$$

$$k^\mu \Gamma^{W^\pm W^\mp}_{\mu\nu}(k,-k) \mp M_W \Gamma^{\phi^\pm W^\mp}_{\nu}(k,-k) = 0 \tag{C.4}$$

$$k^\mu \Gamma^{Z\chi}_{\mu\nu}(k,-k) - iM_Z \Gamma^{\chi\chi}_{\nu}(k,-k) + \frac{ie}{2c_w s_w} T^H = 0 \tag{C.5}$$

$$k^\mu \Gamma^{W^\pm \phi^\mp}_{\mu\nu}(k,-k) \mp M_W \Gamma^{\phi^\pm \phi^\mp}_{\nu}(k,-k) \pm \frac{e}{2s_w} T^H = 0 \tag{C.6}$$

In the previous identities, T^H is the Higgs tadpole contribution. We have found a non-vanishing R_2 contribution to T^H, due to the coupling of H with Z and W, while R_1 does not contribute to T^H.

C.2 Ward identities involving VFF, SFF and FF

$$k^\mu \Gamma^{A\bar{f}f}_{\mu}(k,\bar{p},p) + eQ_f(\Gamma^{\bar{f}f}(\bar{p},k+p) - \Gamma^{\bar{f}f}(k+\bar{p},p)) = 0 \tag{C.7}$$

$$\begin{aligned} k^\mu \Gamma^{Z\bar{f}f}_{\mu}(k,\bar{p},p) - iM_Z \Gamma^{\chi\bar{f}f}(k,\bar{p},p) - e(\Gamma^{\bar{f}f}(\bar{p},k+p)(v_f - a_f\gamma_5) \\ -(v_f + a_f\gamma_5)\Gamma^{\bar{f}f}(k+\bar{p},p)) = 0 \end{aligned} \tag{C.8}$$

[1] We assume $V_{ud} = V^\dagger_{du} = 1$ and understand a sum over colors.

$$k^\mu \Gamma_\mu^{W^+ \bar{f}_u f_d}(k,\bar{p},p) - M_W \Gamma^{\phi^+ \bar{f}_u f_d}(k,\bar{p},p) - \frac{e}{\sqrt{2}s_w}(\Gamma^{\bar{f}_u f_u}(\bar{p},k+p)\Omega_- - \Omega_+ \Gamma^{\bar{f}_d f_d}(k+\bar{p},p)) = 0 \quad \text{(C.9)}$$

$$k^\mu \Gamma_\mu^{W^- \bar{f}_d f_u}(k,\bar{p},p) + M_W \Gamma^{\phi^- \bar{f}_d f_u}(k,\bar{p},p) - \frac{e}{\sqrt{2}s_w}(\Gamma^{\bar{f}_d f_d}(\bar{p},k+p)\Omega_- - \Omega_+ \Gamma^{\bar{f}_u f_u}(k+\bar{p},p)) = 0 \quad \text{(C.10)}$$

In the previous expressions, f_u is a fermion with I_{3f}=1/2, f_d is the fermion of the same weak-isospin doublet with I_{3f}= -1/2, $v_f = (I_{3f} - 2s_w^2 Q_f)/(2s_w c_w)$ and $a_f = I_{3f}/(2s_w c_w)$.

C.3 Ward identities involving VVV, VVS and VV

$$k^\mu \Gamma_{\mu\nu\sigma}^{AW^+W^-}(k,k_+,k_-) - e(\Gamma_{\nu\sigma}^{W^+W^-}(k_+,k+k_-) - \Gamma_{\nu\sigma}^{W^+W^-}(k+k_+,k_-)) = 0 \quad \text{(C.11)}$$

$$k_+^\mu \Gamma_{\mu\nu\sigma}^{W^+W^-A}(k_+,k_-,k) - M_W \Gamma_{\nu\sigma}^{\phi^+W^-A}(k_+,k_-,k) - e\Gamma_{\sigma\nu}^{W^+W^-}(k+k_+,k_-) + e(\Gamma_{\sigma\nu}^{AA}(k,k_++k_-) - \frac{c_w}{s_w}\Gamma_{\sigma\nu}^{AZ}(k,k_++k_-)) = 0 \quad \text{(C.12)}$$

$$k_-^\mu \Gamma_{\mu\nu\sigma}^{W^-W^+A}(k_-,k_+,k) + M_W \Gamma_{\nu\sigma}^{\phi^-W^+A}(k_-,k_+,k) + e\Gamma_{\sigma\nu}^{W^-W^+}(k+k_-,k_+) - e(\Gamma_{\sigma\nu}^{AA}(k,k_++k_-) - \frac{c_w}{s_w}\Gamma_{\sigma\nu}^{AZ}(k,k_++k_-)) = 0 \quad \text{(C.13)}$$

$$k^\mu \Gamma_{\mu\nu\sigma}^{ZW^+W^-}(k,k_+,k_-) - iM_Z \Gamma_{\nu\sigma}^{\chi W^+W^-}(k,k_+,k_-) - e\frac{c_w}{s_w}(\Gamma_{\nu\sigma}^{W^+W^-}(k+k_+,k_-) - \Gamma_{\sigma\nu}^{W^-W^+}(k+k_-,k_+)) = 0 \quad \text{(C.14)}$$

$$k_+^\mu \Gamma^{W^+W^-Z}_{\mu\nu\sigma}(k_+,k_-,k) - M_W \Gamma^{\phi^+W^-Z}_{\nu\sigma}(k_+,k_-,k) + e\frac{c_w}{s_w}\Gamma^{W^+W^-}_{\sigma\nu}(k+k_+,k_-) \\ +e(\Gamma^{AZ}_{\nu\sigma}(k_++k_-,k) - \frac{c_w}{s_w}\Gamma^{ZZ}_{\nu\sigma}(k_++k_-,k)) = 0 \tag{C.15}$$

$$k_-^\mu \Gamma^{W^-W^+Z}_{\mu\nu\sigma}(k_-,k_+,k) + M_W \Gamma^{\phi^-W^+Z}_{\nu\sigma}(k_-,k_+,k) - e\frac{c_w}{s_w}\Gamma^{W^-W^+}_{\sigma\nu}(k+k_-,k_+) \\ -e(\Gamma^{AZ}_{\nu\sigma}(k_++k_-,k) - \frac{c_w}{s_w}\Gamma^{ZZ}_{\nu\sigma}(k_++k_-,k)) = 0 \tag{C.16}$$

C.4 Ward identities involving VVS, VSS and VS

$$k_1^\mu \Gamma^{AAH}_{\mu\nu}(k_1,k_2,k_3) = k_1^\mu \Gamma^{AA\chi}_{\mu\nu}(k_1,k_2,k_3) \\ = k_1^\mu \Gamma^{AZH}_{\mu\nu}(k_1,k_2,k_3) = k_1^\mu \Gamma^{AZ\chi}_{\mu\nu}(k_1,k_2,k_3) = 0 \tag{C.17}$$

$$k^\mu \Gamma^{AW^+\phi^-}_{\mu\nu}(k,k_+,k_-) + e\Gamma^{W^+\phi^-}_{\nu}(k+k_+,k_-) - e\Gamma^{\phi^-W^+}_{\nu}(k+k_-,k_+) = 0 \tag{C.18}$$

$$k^\mu \Gamma^{AW^-\phi^+}_{\mu\nu}(k,k_-,k_+) - e\Gamma^{W^-\phi^+}_{\nu}(k+k_-,k_+) + e\Gamma^{\phi^+W^-}_{\nu}(k+k_+,k_-) = 0 \tag{C.19}$$

$$k_1^\mu \Gamma^{ZAH}_{\mu\nu}(k_1,k_2,k_3) - iM_Z \Gamma^{\chi AH}_{\nu}(k_1,k_2,k_3) - \frac{ie}{2c_w s_w}\Gamma^{\chi A}_{\nu}(k_1+k_3,k_2) = 0 \tag{C.20}$$

$$k_1^\mu \Gamma_{\mu\nu}^{ZA\chi}(k_1,k_2,k_3) - iM_Z\Gamma_\nu^{\chi A\chi}(k_1,k_2,k_3) + \frac{ie}{2c_w s_w}\Gamma_\nu^{HA}(k_1+k_3,k_2) = 0 \tag{C.21}$$

$$k_1^\mu \Gamma_{\mu\nu}^{ZZH}(k_1,k_2,k_3) - iM_Z\Gamma_\nu^{\chi ZH}(k_1,k_2,k_3) - \frac{ie}{2c_w s_w}\Gamma_\nu^{\chi Z}(k_1+k_3,k_2) = 0 \tag{C.22}$$

$$k_1^\mu \Gamma_{\mu\nu}^{ZZ\chi}(k_1,k_2,k_3) - iM_Z\Gamma_\nu^{\chi Z\chi}(k_1,k_2,k_3) + \frac{ie}{2c_w s_w}\Gamma_\nu^{HZ}(k_1+k_3,k_2) = 0 \tag{C.23}$$

$$\begin{aligned} k^\mu \Gamma_{\mu\nu}^{ZW^+\phi^-}(k,k_+,k_-) - iM_Z\Gamma_\nu^{\chi W^+\phi^-}(k,k_+,k_-) \\ -e\frac{c_w}{s_w}\Gamma_\nu^{W^+\phi^-}(k+k_+,k_-) + e\frac{c_w^2-s_w^2}{2c_w s_w}\Gamma_\nu^{\phi^- W^+}(k+k_-,k_+) = 0 \end{aligned} \tag{C.24}$$

$$\begin{aligned} k^\mu \Gamma_{\mu\nu}^{ZW^-\phi^+}(k,k_-,k_+) - iM_Z\Gamma_\nu^{\chi W^-\phi^+}(k,k_-,k_+) \\ +e\frac{c_w}{s_w}\Gamma_\nu^{W^-\phi^+}(k+k_-,k_+) - e\frac{c_w^2-s_w^2}{2c_w s_w}\Gamma_\nu^{\phi^+ W^-}(k+k_+,k_-) = 0 \end{aligned} \tag{C.25}$$

$$\begin{aligned} k_+^\mu \Gamma_{\mu\nu}^{W^+A\phi^-}(k_+,k,k_-) - M_W\Gamma_\nu^{\phi^+A\phi^-}(k_+,k,k_-) \\ -e\Gamma_\nu^{W^+\phi^-}(k+k_+,k_-) + \frac{e}{2s_w}(\Gamma_\nu^{HA}(k_++k_-,k) + i\Gamma_\nu^{\chi A}(k_++k_-,k)) = 0 \end{aligned} \tag{C.26}$$

$$
\begin{aligned}
& k_+^\mu \Gamma_{\mu\nu}^{W^+ Z\phi^-}(k_+, k, k_-) - M_W \Gamma_\nu^{\phi^+ Z\phi^-}(k_+, k, k_-) \\
& + e\frac{c_w}{s_w}\Gamma_\nu^{W^+\phi^-}(k + k_+, k_-) + \frac{e}{2s_w}(\Gamma_\nu^{HZ}(k_+ + k_-, k) + i\Gamma_\nu^{\chi Z}(k_+ + k_-, k)) = 0
\end{aligned}
\tag{C.27}
$$

$$
\begin{aligned}
& k_+^\mu \Gamma_{\mu\nu}^{W^+ W^- H}(k_+, k_-, k) - M_W \Gamma_\nu^{\phi^+ W^- H}(k_+, k_-, k) \\
& - \frac{e}{2s_w}\Gamma_\nu^{\phi^+ W^-}(k + k_+, k_-) + e(\Gamma_\nu^{AH}(k_+ + k_-, k) - \frac{c_w}{s_w}\Gamma_\nu^{ZH}(k_+ + k_-, k)) = 0
\end{aligned}
\tag{C.28}
$$

$$
\begin{aligned}
& k_+^\mu \Gamma_{\mu\nu}^{W^+ W^- \chi}(k_+, k_-, k) - M_W \Gamma_\nu^{\phi^+ W^- \chi}(k_+, k_-, k) \\
& - \frac{ie}{2s_w}\Gamma_\nu^{\phi^+ W^-}(k + k_+, k_-) + e(\Gamma_\nu^{A\chi}(k_+ + k_-, k) - \frac{c_w}{s_w}\Gamma_\nu^{Z\chi}(k_+ + k_-, k)) = 0
\end{aligned}
\tag{C.29}
$$

$$
\begin{aligned}
& k_-^\mu \Gamma_{\mu\nu}^{W^- A\phi^+}(k_-, k, k_+) + M_W \Gamma_\nu^{\phi^- A\phi^+}(k_-, k, k_+) \\
& + e\Gamma_\nu^{W^-\phi^+}(k + k_-, k_+) - \frac{e}{2s_w}(\Gamma_\nu^{HA}(k_+ + k_-, k) - i\Gamma_\nu^{\chi A}(k_+ + k_-, k)) = 0
\end{aligned}
\tag{C.30}
$$

$$
\begin{aligned}
& k_-^\mu \Gamma_{\mu\nu}^{W^- Z\phi^+}(k_-, k, k_+) + M_W \Gamma_\nu^{\phi^- Z\phi^+}(k_-, k, k_+) \\
& - e\frac{c_w}{s_w}\Gamma_\nu^{W^-\phi^+}(k + k_-, k_+) - \frac{e}{2s_w}(\Gamma_\nu^{HZ}(k_+ + k_-, k) - i\Gamma_\nu^{\chi Z}(k_+ + k_-, k)) = 0
\end{aligned}
\tag{C.31}
$$

$$
\begin{aligned}
& k_-^\mu \Gamma_{\mu\nu}^{W^- W^+ H}(k_-, k_+, k) + M_W \Gamma_\nu^{\phi^- W^+ H}(k_-, k_+, k) \\
& + \frac{e}{2s_w}\Gamma_\nu^{\phi^- W^+}(k + k_-, k_+) - e(\Gamma_\nu^{AH}(k_+ + k_-, k) - \frac{c_w}{s_w}\Gamma_\nu^{ZH}(k_+ + k_-, k)) = 0
\end{aligned}
\tag{C.32}
$$

$$
\begin{aligned}
& k_-^{\mu}\Gamma_{\mu\nu}^{W^-W^+\chi}(k_-,k_+,k) + M_W\Gamma_{\nu}^{\phi^-W^+\chi}(k_-,k_+,k) \\
-\frac{ie}{2s_w}\Gamma_{\nu}^{\phi^-W^+}(k+k_-,k_+) - e(\Gamma_{\nu}^{A\chi}(k_++k_-,k) - \frac{c_w}{s_w}\Gamma_{\nu}^{Z\chi}(k_++k_-,k)) = 0
\end{aligned}
\tag{C.33}
$$

C.5 Ward identities involving VSS, SSS and SS

$$
k_1^{\mu}\Gamma_{\mu}^{AHH}(k_1,k_2,k_3) = k_1^{\mu}\Gamma_{\mu}^{AH\chi}(k_1,k_2,k_3) = k_1^{\mu}\Gamma_{\mu}^{A\chi\chi}(k_1,k_2,k_3) = 0 \tag{C.34}
$$

$$
k^{\mu}\Gamma_{\mu}^{A\phi^+\phi^-}(k,k_+,k_-) + e(\Gamma^{\phi^+\phi^-}(k+k_+,k_-) - \Gamma^{\phi^-\phi^+}(k+k_-,k_+)) = 0 \tag{C.35}
$$

$$
\begin{aligned}
k_1^{\mu}\Gamma_{\mu}^{ZHH}(k_1,k_2,k_3) - iM_Z\Gamma^{\chi HH}(k_1,k_2,k_3) \quad &- \quad \frac{ie}{2c_ws_w}\Big(\Gamma^{\chi H}(k_1+k_2,k_3) \\
&+ \quad \Gamma^{\chi H}(k_1+k_3,k_2)\Big) = 0
\end{aligned}
\tag{C.36}
$$

$$
\begin{aligned}
k_1^{\mu}\Gamma_{\mu}^{ZH\chi}(k_1,k_2,k_3) - iM_Z\Gamma^{\chi H\chi}(k_1,k_2,k_3) \quad &- \quad \frac{ie}{2c_ws_w}\left(\Gamma^{\chi\chi}(k_1+k_2,k_3)\right. \\
&- \quad \Gamma^{HH}(k_1+k_3,k_2)\Big) = 0
\end{aligned}
\tag{C.37}
$$

$$
\begin{aligned}
k_1^{\mu}\Gamma_{\mu}^{Z\chi\chi}(k_1,k_2,k_3) - iM_Z\Gamma^{\chi\chi\chi}(k_1,k_2,k_3) \quad &+ \quad \frac{ie}{2c_ws_w}\Big(\Gamma^{H\chi}(k_1+k_2,k_3) \\
&+ \quad \Gamma^{H\chi}(k_1+k_3,k_2)\Big) = 0
\end{aligned}
\tag{C.38}
$$

$$k^\mu \Gamma_\mu^{Z\phi^+\phi^-}(k,k_+,k_-) - iM_Z\Gamma^{\chi\phi^+\phi^-}(k,k_+,k_-) \\ -e\frac{c_w^2 - s_w^2}{2c_w s_w}(\Gamma^{\phi^+\phi^-}(k+k_+,k_-) - \Gamma^{\phi^-\phi^+}(k+k_-,k_+)) = 0 \tag{C.39}$$

$$k_+^\mu \Gamma_\mu^{W^+H\phi^-}(k_+,k,k_-) - M_W\Gamma^{\phi^+H\phi^-}(k_+,k,k_-) \\ +\frac{e}{2s_w}(\Gamma^{HH}(k_-+k_+,k) + i\Gamma^{\chi H}(k_++k_-,k)) - \frac{e}{2s_w}\Gamma^{\phi^+\phi^-}(k_++k,k_-) = 0 \tag{C.40}$$

$$k_+^\mu \Gamma_\mu^{W^+\chi\phi^-}(k_+,k,k_-) - M_W\Gamma^{\phi^+\chi\phi^-}(k_+,k,k_-) \\ +\frac{e}{2s_w}(\Gamma^{H\chi}(k_-+k_+,k) + i\Gamma^{\chi\chi}(k_++k_-,k)) - \frac{ie}{2s_w}\Gamma^{\phi^+\phi^-}(k_++k,k_-) = 0 \tag{C.41}$$

$$k_-^\mu \Gamma_\mu^{W^-H\phi^+}(k_-,k,k_+) + M_W\Gamma^{\phi^-H\phi^+}(k_-,k,k_+) \\ -\frac{e}{2s_w}(\Gamma^{HH}(k_-+k_+,k) - i\Gamma^{\chi H}(k_++k_-,k)) + \frac{e}{2s_w}\Gamma^{\phi^-\phi^+}(k_-+k,k_+) = 0 \tag{C.42}$$

$$k_-^\mu \Gamma_\mu^{W^-\chi\phi^+}(k_-,k,k_+) + M_W\Gamma^{\phi^-\chi\phi^+}(k_-,k,k_+) \\ -\frac{e}{2s_w}(\Gamma^{H\chi}(k_-+k_+,k) - i\Gamma^{\chi\chi}(k_++k_-,k)) - \frac{ie}{2s_w}\Gamma^{\phi^-\phi^+}(k_-+k,k_+) = 0 \tag{C.43}$$

C.6 Ward identities involving VVVV, VVVS and VVV

$$k_{1,2,3,4}^\mu \Gamma_{\mu\nu\kappa\sigma}^{AAAA}(k_1,k_2,k_3,k_4) = 0 \tag{C.44}$$

$$k^{\mu}_{1,2,3}\Gamma^{AAAZ}_{\mu\nu\kappa\sigma}(k_1,k_2,k_3,k_4)=0 \tag{C.45}$$

$$k^{\mu}_{1,2}\Gamma^{AAZZ}_{\mu\nu\kappa\sigma}(k_1,k_2,k_3,k_4)=0 \tag{C.46}$$

$$k^{\mu}_{1}\Gamma^{AZZZ}_{\mu\nu\kappa\sigma}(k_1,k_2,k_3,k_4)=0 \tag{C.47}$$

$$\begin{aligned} k^{\mu}_{1}\Gamma^{AAW^+W^-}_{\mu\nu\kappa\sigma}(k_1,k_2,k_+,k_-)+ \quad & e \quad \Big[\Gamma^{AW^+W^-}_{\nu\kappa\sigma}(k_2,k_1+k_+,k_-) \\ & - \quad \Gamma^{AW^+W^-}_{\nu\kappa\sigma}(k_2,k_+,k_1+k_-)\Big]=0 \end{aligned} \tag{C.48}$$

$$\begin{aligned} k^{\mu}_{1}\Gamma^{AZW^+W^-}_{\mu\nu\kappa\sigma}(k_1,k_2,k_+,k_-)+ \quad & e \quad \Big[\Gamma^{W^+ZW^-}_{\kappa\nu\sigma}(k_1+k_+,k_2,k_-) \\ & - \quad \Gamma^{W^-ZW^+}_{\sigma\nu\kappa}(k_1+k_-,k_2,k_+)\Big]=0 \end{aligned} \tag{C.49}$$

$$k^{\mu}\Gamma^{ZV_2V_3V_4}_{\mu\nu\kappa\sigma}(k_1,k_2,k_3,k_4)-iM_Z\Gamma^{\chi V_2V_3V_4}_{\nu\kappa\sigma}(k_1,k_2,k_3,k_4)=0\,, \tag{C.50}$$

where k here refers to any of the Z momenta and V's stand for A or Z.

$$\begin{aligned} k^{\mu}_{+} \quad & \Gamma^{W^+W^-AA}_{\mu\nu\kappa\sigma}(k_+,k_-,k_3,k_4)+e\Big[\Gamma^{AAA}_{\nu\kappa\sigma}(k_++k_-,k_3,k_4)-\frac{c_w}{s_w}\Gamma^{ZAA}_{\nu\kappa\sigma}(k_++k_-,k_3,k_4) \\ - \quad & \Gamma^{W^+W^-A}_{\sigma\nu\kappa}(k_++k_4,k_-,k_3)-\Gamma^{W^-W^+A}_{\nu\kappa\sigma}(k_-,k_++k_3,k_4)\Big] \\ - \quad & M_W\Gamma^{\phi^+W^-AA}_{\nu\kappa\sigma}(k_+,k_-,k_3,k_4)=0 \end{aligned} \tag{C.51}$$

$$\begin{aligned}
k_-^\mu \quad & \Gamma^{W^-W^+AA}_{\mu\nu\kappa\sigma}(k_-,k_+,k_3,k_4) - e\Big[\Gamma^{AAA}_{\nu\kappa\sigma}(k_+ + k_-,k_3,k_4) - \frac{c_w}{s_w}\Gamma^{ZAA}_{\nu\kappa\sigma}(k_+ + k_-,k_3,k_4) \\
- \quad & \Gamma^{W^-W^+A}_{\sigma\nu\kappa}(k_- + k_4,k_+,k_3) - \Gamma^{W^+W^-A}_{\nu\kappa\sigma}(k_+,k_- + k_3,k_4)\Big] \\
+ \quad & M_W\Gamma^{\phi^-W^+AA}_{\nu\kappa\sigma}(k_-,k_+,k_3,k_4) = 0
\end{aligned} \tag{C.52}$$

$$\begin{aligned}
k_1^\mu \quad & \Gamma^{ZZW^+W^-}_{\mu\nu\kappa\sigma}(k_1,k_2,k_+,k_-) - e\frac{c_w}{s_w}\Big[\Gamma^{W^+ZW^-}_{\kappa\nu\sigma}(k_1 + k_+,k_2,k_-) \\
- \quad & \Gamma^{W^-ZW^+}_{\sigma\nu\kappa}(k_1 + k_-,k_2,k_+)\Big] - iM_Z\Gamma^{\chi ZW^+W^-}_{\nu\kappa\sigma}(k_1,k_2,k_+,k_-) = 0
\end{aligned} \tag{C.53}$$

$$\begin{aligned}
k_+^\mu \quad & \Gamma^{W^+ZZW^-}_{\mu\nu\kappa\sigma}(k_+,k_1,k_2,k_-) + e\Big[\Gamma^{AZZ}_{\sigma\nu\kappa}(k_+ + k_-,k_1,k_2) - \frac{c_w}{s_w}\Gamma^{ZZZ}_{\sigma\nu\kappa}(k_+ + k_-,k_1,k_2) \\
+ \quad & \frac{c_w}{s_w}\Gamma^{W^+ZW^-}_{\nu\kappa\sigma}(k_+ + k_1,k_2,k_-) + \frac{c_w}{s_w}\Gamma^{W^+ZW^-}_{\kappa\nu\sigma}(k_+ + k_2,k_1,k_-)\Big] \\
- \quad & M_W\Gamma^{\phi^+ZZW^-}_{\nu\kappa\sigma}(k_+,k_1,k_2,k_-) = 0
\end{aligned} \tag{C.54}$$

$$\begin{aligned}
k_-^\mu \quad & \Gamma^{W^-W^+ZZ}_{\mu\nu\kappa\sigma}(k_-,k_+,k_3,k_4) - e\Big[\Gamma^{AZZ}_{\nu\kappa\sigma}(k_+ + k_-,k_3,k_4) - \frac{c_w}{s_w}\Gamma^{ZZZ}_{\nu\kappa\sigma}(k_+ + k_-,k_3,k_4) \\
+ \quad & \frac{c_w}{s_w}\Gamma^{W^-W^+Z}_{\kappa\nu\sigma}(k_- + k_3,k_+,k_4) + \frac{c_w}{s_w}\Gamma^{W^-W^+Z}_{\sigma\nu\kappa}(k_- + k_4,k_+,k_3)\Big] \\
+ \quad & M_W\Gamma^{\phi^-W^+ZZ}_{\nu\kappa\sigma}(k_-,k_+,k_3,k_4) = 0
\end{aligned} \tag{C.55}$$

$$\begin{aligned}
k_1^\mu \quad & \Gamma^{ZAW^+W^-}_{\mu\nu\kappa\sigma}(k_1,k_2,k_+,k_-) - e\frac{c_w}{s_w}\Big[\Gamma^{W^+AW^-}_{\kappa\nu\sigma}(k_1 + k_+,k_2,k_-) \\
- \quad & \Gamma^{W^-AW^+}_{\sigma\nu\kappa}(k_1 + k_-,k_2,k_+)\Big] - iM_Z\Gamma^{\chi AW^+W^-}_{\nu\kappa\sigma}(k_1,k_2,k_+,k_-) = 0
\end{aligned} \tag{C.56}$$

$$\begin{aligned}
k_+^\mu \quad & \Gamma^{W^+W^-AZ}_{\mu\nu\kappa\sigma}(k_+,k_-,k_3,k_4) + e\Big[\Gamma^{AAZ}_{\nu\kappa\sigma}(k_++k_-,k_3,k_4) - \frac{c_w}{s_w}\Gamma^{ZAZ}_{\nu\kappa\sigma}(k_++k_-,k_3,k_4) \\
- \quad & \Gamma^{W^+W^-Z}_{\kappa\nu\sigma}(k_++k_3,k_-,k_4) + \frac{c_w}{s_w}\Gamma^{W^+W^-A}_{\sigma\nu\kappa}(k_++k_4,k_-,k_3)\Big] \\
- \quad & M_W\Gamma^{\phi^+W^-AZ}_{\nu\kappa\sigma}(k_+,k_-,k_3,k_4) = 0
\end{aligned} \tag{C.57}$$

$$\begin{aligned}
k_-^\mu \quad & \Gamma^{W^-W^+AZ}_{\mu\nu\kappa\sigma}(k_-,k_+,k_3,k_4) - e\Big[\Gamma^{AAZ}_{\nu\kappa\sigma}(k_++k_-,k_3,k_4) - \frac{c_w}{s_w}\Gamma^{ZAZ}_{\nu\kappa\sigma}(k_++k_-,k_3,k_4) \\
- \quad & \Gamma^{W^-W^+Z}_{\kappa\nu\sigma}(k_-+k_3,k_+,k_4) + \frac{c_w}{s_w}\Gamma^{W^-W^+A}_{\sigma\nu\kappa}(k_-+k_4,k_+,k_3)\Big] \\
+ \quad & M_W\Gamma^{\phi^-W^+AZ}_{\nu\kappa\sigma}(k_-,k_+,k_3,k_4) = 0
\end{aligned} \tag{C.58}$$

$$\begin{aligned}
k_{1+}^\mu \quad & \Gamma^{W^+W^-W^+W^-}_{\mu\nu\kappa\sigma}(k_{1+},k_{1-},k_{2+},k_{2-}) + e\Big[\Gamma^{AW^+W^-}_{\nu\kappa\sigma}(k_{1+}+k_{1-},k_{2+},k_{2-}) \\
- \quad & \frac{c_w}{s_w}\Gamma^{ZW^+W^-}_{\nu\kappa\sigma}(k_{1+}+k_{1-},k_{2+},k_{2-}) + \Gamma^{AW^-W^+}_{\sigma\nu\kappa}(k_{1+}+k_{2-},k_{1-},k_{2+}) \\
- \quad & \frac{c_w}{s_w}\Gamma^{ZW^-W^+}_{\sigma\nu\kappa}(k_{1+}+k_{2-},k_{1-},k_{2+})\Big] - M_W\Gamma^{\phi^+W^-W^+W^-}_{\nu\kappa\sigma}(k_{1+},k_{1-},k_{2+},k_{2-}) = 0
\end{aligned} \tag{C.59}$$

$$\begin{aligned}
k_{1-}^\mu \quad & \Gamma^{W^-W^+W^-W^+}_{\mu\nu\kappa\sigma}(k_{1-},k_{1+},k_{2-},k_{2+}) - e\Big[\Gamma^{AW^-W^+}_{\nu\kappa\sigma}(k_{1+}+k_{1-},k_{2-},k_{2+}) \\
- \quad & \frac{c_w}{s_w}\Gamma^{ZW^-W^+}_{\nu\kappa\sigma}(k_{1+}+k_{1-},k_{2-},k_{2+}) + \Gamma^{AW^+W^-}_{\sigma\nu\kappa}(k_{2+}+k_{1-},k_{1+},k_{2-}) \\
- \quad & \frac{c_w}{s_w}\Gamma^{ZW^+W^-}_{\sigma\nu\kappa}(k_{2+}+k_{1-},k_{1+},k_{2-})\Big] + M_W\Gamma^{\phi^-W^+W^-W^+}_{\nu\kappa\sigma}(k_{1-},k_{1+},k_{2-},k_{2+}) = 0
\end{aligned} \tag{C.60}$$

C.7 Ward identities involving SSSS, VSSS and SSS

$$\begin{aligned} k_1^\mu &\quad \Gamma_\mu^{Z\chi HH}(k_1,k_2,k_3,k_4) - iM_Z\Gamma^{\chi\chi HH}(k_1,k_2,k_3,k_4) - \frac{ie}{2c_ws_w}\Big[\Gamma^{\chi\chi H}(k_1+k_3,k_2,k_4) \\ + &\quad \Gamma^{\chi\chi H}(k_1+k_4,k_2,k_3) - \Gamma^{HHH}(k_1+k_2,k_3,k_4)\Big] = 0 \end{aligned} \tag{C.61}$$

$$\begin{aligned} k_1^\mu &\quad \Gamma_\mu^{Z\chi\chi\chi}(k_1,k_2,k_3,k_4) - iM_Z\Gamma^{\chi\chi\chi\chi}(k_1,k_2,k_3,k_4) + \frac{ie}{2c_ws_w}\Big[\Gamma^{H\chi\chi}(k_1+k_2,k_3,k_4) \\ + &\quad \Gamma^{H\chi\chi}(k_1+k_3,k_2,k_4) + \Gamma^{H\chi\chi}(k_1+k_4,k_2,k_3)\Big] = 0 \end{aligned} \tag{C.62}$$

$$\begin{aligned} k_1^\mu &\quad \Gamma_\mu^{ZH\phi^+\phi^-}(k_1,k_2,k_+,k_-) - iM_Z\Gamma^{\chi H\phi^+\phi^-}(k_1,k_2,k_+,k_-) \\ - &\quad e\frac{c_w^2-s_w^2}{2c_ws_w}\Big[\Gamma^{\phi^+H\phi^-}(k_1+k_+,k_2,k_-) - \Gamma^{\phi^-H\phi^+}(k_1+k_-,k_2,k_+)\Big] \\ - &\quad \frac{ie}{2c_ws_w}\Gamma^{\chi\phi^+\phi^-}(k_1+k_2,k_+,k_-) = 0 \end{aligned} \tag{C.63}$$

$$\begin{aligned} k_1^\mu &\quad \Gamma_\mu^{Z\chi\phi^+\phi^-}(k_1,k_2,k_+,k_-) - iM_Z\Gamma^{\chi\chi\phi^+\phi^-}(k_1,k_2,k_+,k_-) \\ - &\quad e\frac{c_w^2-s_w^2}{2c_ws_w}\Big[\Gamma^{\phi^+\chi\phi^-}(k_1+k_+,k_2,k_-) - \Gamma^{\phi^-\chi\phi^+}(k_1+k_-,k_2,k_+)\Big] \\ + &\quad \frac{ie}{2c_ws_w}\Gamma^{H\phi^+\phi^-}(k_1+k_2,k_+,k_-) = 0 \end{aligned} \tag{C.64}$$

$$\begin{aligned} k_+^\mu &\quad \Gamma_\mu^{W^+\phi^-HH}(k_+,k_-,k_1,k_2) - M_W\Gamma^{\phi^+\phi^-HH}(k_+,k_-,k_1,k_2) \\ + &\quad \frac{e}{2s_w}\Big[\Gamma^{HHH}(k_++k_-,k_1,k_2) + i\Gamma^{\chi HH}(k_++k_-,k_1,k_2) \\ - &\quad \Gamma^{\phi^+\phi^-H}(k_1+k_+,k_-,k_2) - \Gamma^{\phi^+\phi^-H}(k_2+k_+,k_-,k_1)\Big] = 0 \end{aligned} \tag{C.65}$$

$$\begin{aligned} k_+^\mu \quad & \Gamma_\mu^{W^+\phi^- H\chi}(k_+, k_-, k_1, k_2) - M_W \Gamma^{\phi^+\phi^- H\chi}(k_+, k_-, k_1, k_2) + \frac{e}{2s_w}\Big[\Gamma^{HH\chi}(k_+ + k_-, k_1, k_2) \\ + \quad & i\Gamma^{\chi H\chi}(k_+ + k_-, k_1, k_2) - \Gamma^{\phi^+\phi^-\chi}(k_1 + k_+, k_-, k_2) - i\Gamma^{\phi^+\phi^- H}(k_2 + k_+, k_-, k_1)\Big] = 0 \end{aligned} \tag{C.66}$$

$$\begin{aligned} k_+^\mu \quad & \Gamma_\mu^{W^+\phi^-\chi\chi}(k_+, k_-, k_1, k_2) - M_W \Gamma^{\phi^+\phi^-\chi\chi}(k_+, k_-, k_1, k_2) \\ + \quad & \frac{e}{2s_w}\Big[\Gamma^{H\chi\chi}(k_+ + k_-, k_1, k_2) + i\Gamma^{\chi\chi\chi}(k_+ + k_-, k_1, k_2) \\ - \quad & i\,\Gamma^{\phi^+\phi^-\chi}(k_1 + k_+, k_-, k_2) - i\Gamma^{\phi^+\phi^-\chi}(k_2 + k_+, k_-, k_1)\Big] = 0 \end{aligned} \tag{C.67}$$

$$\begin{aligned} k_{1+}^\mu \quad & \Gamma_\mu^{W^+\phi^-\phi^+\phi^-}(k_{1+}, k_{1-}, k_{2+}, k_{2-}) - M_W \Gamma^{\phi^+\phi^-\phi^+\phi^-}(k_{1+}, k_{1-}, k_{2+}, k_{2-}) \\ + \quad & \frac{e}{2s_w}\Big[\Gamma^{H\phi^+\phi^-}(k_{1+} + k_{1-}, k_{2+}, k_{2-}) + i\Gamma^{\chi\phi^+\phi^-}(k_{1+} + k_{1-}, k_{2+}, k_{2-}) \\ + \quad & \Gamma^{H\phi^-\phi^+}(k_{1+} + k_{2-}, k_{1-}, k_{2+}) + i\Gamma^{\chi\phi^-\phi^+}(k_{1+} + k_{2-}, k_{1-}, k_{2+})\Big] = 0 \end{aligned} \tag{C.68}$$

$$\begin{aligned} k_-^\mu \quad & \Gamma_\mu^{W^-\phi^+ HH}(k_-, k_+, k_1, k_2) + M_W \Gamma^{\phi^-\phi^+ HH}(k_-, k_+, k_1, k_2) \\ - \quad & \frac{e}{2s_w}\Big[\Gamma^{HHH}(k_+ + k_-, k_1, k_2) - i\Gamma^{\chi HH}(k_+ + k_-, k_1, k_2) \\ - \quad & \Gamma^{\phi^-\phi^+ H}(k_1 + k_-, k_+, k_2) - \Gamma^{\phi^-\phi^+ H}(k_2 + k_-, k_+, k_1)\Big] = 0 \end{aligned} \tag{C.69}$$

$$\begin{aligned} k_-^\mu \quad & \Gamma_\mu^{W^-\phi^+ H\chi}(k_-, k_+, k_1, k_2) + M_W \Gamma^{\phi^-\phi^+ H\chi}(k_-, k_+, k_1, k_2) \\ - \quad & \frac{e}{2s_w}\Big[\Gamma^{HH\chi}(k_+ + k_-, k_1, k_2) - i\Gamma^{\chi H\chi}(k_+ + k_-, k_1, k_2) \\ - \quad & \Gamma^{\phi^-\phi^+\chi}(k_1 + k_-, k_+, k_2) + i\Gamma^{\phi^-\phi^+ H}(k_2 + k_-, k_+, k_1)\Big] = 0 \end{aligned} \tag{C.70}$$

$$\begin{aligned}
k_-^\mu & \Gamma_\mu^{W^-\phi^+\chi\chi}(k_-,k_+,k_1,k_2) + M_W \Gamma^{\phi^-\phi^+\chi\chi}(k_-,k_+,k_1,k_2) \\
- & \frac{e}{2s_w}\Big[\Gamma^{H\chi\chi}(k_+ + k_-,k_1,k_2) - i\Gamma^{\chi\chi\chi}(k_+ + k_-,k_1,k_2) \\
+ & i\,\Gamma^{\phi^-\phi^+\chi}(k_1 + k_-,k_+,k_2) + i\Gamma^{\phi^-\phi^+\chi}(k_2 + k_-,k_+,k_1)\Big] = 0
\end{aligned} \tag{C.71}$$

$$\begin{aligned}
k_{1-}^\mu & \Gamma_\mu^{W^-\phi^+\phi^-\phi^+}(k_{1-},k_{1+},k_{2-},k_{2+}) + M_W \Gamma^{\phi^-\phi^+\phi^-\phi^+}(k_{1-},k_{1+},k_{2-},k_{2+}) \\
- & \frac{e}{2s_w}\Big[\Gamma^{H\phi^-\phi^+}(k_{1+} + k_{1-},k_{2-},k_{2+}) - i\Gamma^{\chi\phi^-\phi^+}(k_{1+} + k_{1-},k_{2-},k_{2+}) \\
+ & \Gamma^{H\phi^+\phi^-}(k_{2+} + k_{1-},k_{1+},k_{2-}) - i\Gamma^{\chi\phi^+\phi^-}(k_{2+} + k_{1-},k_{1+},k_{2-})\Big] = 0
\end{aligned} \tag{C.72}$$

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Summary

We now move to the summary of this thesis. During the thesis, there were a lot of subjects considered, concerning both one and two loop calculations. The starting point of the thesis was the OPP method [15]. The authors there, having in mind that any one loop amplitude is expressed as a linear combination of scalar Feynman Integrals with up to four denominators in four dimensions, they wrote a master formula expressing the numerator of the amplitude in terms of products of inverse propagators times some coefficients. The coefficients were either constant in the loop momentum, either had specific forms such that the resulting integrals would vanish upon integration. In other terms, since it was known that any one loop Feynman integral is a linear combinations of other integrals, they found the missing terms to extend this equality at the integrand level. The missing terms are spurious and thus don't exist at the integral level.

Another important part of the OPP method is the calculation of the coefficients that multiply these products of propagators. The difficulty of an actual one loop calculation became actually an algebraic problem of finding these coefficients. Once obtained, one has to multiply these coefficients with the values of a limited number of integrals and get the result. Of course, the success of the one loop scheme is that these integrals have been calculated in the past and automated numerical programs for this purpose already existed in the market. The way of calculating the coefficients from the OPP master formula is with the use of unitarity based cuts. By finding values of the loop momentum where the inverse propagators become zero, the algebraic system becomes triangular and is solved more easily. The fact that one can actually use these values is guaranteed by the fact that the method works at the integrand level. By cutting all possible sets of four,three,two and one propagators, one can find recursively all d, c, b and a type of coefficients (spurious and non-spurious) respectively. Since the OPP method was the base of the thesis, we presented all the features above extensively. In chapter two we show explicitely the OPP master formula and how one can calculate the coefficients of the four, three, two and one point functions. We notice here that the way we presented the spurious terms differs from the one presented in [15]. There, the coefficients are presented in a way more suitable for a numerical implementation of the OPP method, in our case we tried to present them in a way that is more understandable and general.

Having the OPP method in mind we first of all tried to understand why is it possible to write such a master formula at the integrand level in the first place. The motivation behind that reason is that although at one loop the base of Feynman integrals was known in advance, we wanted at some point to extend the method in higher loops. To make this point clear, when writing such a master formula, one basically writes a number of algebraic equations. Both the numerator and the product of denominators in the left and the right hand side of these equations are polynomials that should be equal term by term. We explored by modifying the coefficients and their dependence in the loop momentum, when is it possible to actually solve these systems by comparing number of independent coefficients and tensor structures that can in principle be constructed. Without loss of generality (as we proved in chapter one) we focused on scalar integrals. We proved that if the coefficients have linear dependence in the loop momentum, then every integral can be written in terms of integrals up to four denominators in four dimensions. Actually, we generalised this result in all integer values of dimensions and we saw that linear terms are sufficient to achieve reductions up to integrals with d denominators in d dimensions. With a simple argument, we proved that one cannot decompose an integral more in this way (at one loop never).

Our reduction is not the OPP method. We didn't assume any spurious terms, we just understood why the OPP method needs these spurious terms to work. We also proved that the two reductions are connected and we gave examples where we showed how one can get the OPP reduction from ours, proving that the actual result is unique. The spurious terms have specific properties allowing for less tensor structures to be constructed giving a reduction with a minimal number of coefficients. We also showed that explicitely in the Appendix A, completing the proof of the equivalence of both reductions.

In the first two chapters we focused on reduction methods in four (or d where d is integer) dimensions. However, in most cases, loop integrals suffer from divergencies and dimensional regularisation is used to regulate them. That means that actual calculations are performed in $4+\epsilon$ (or $4-2\epsilon$ in other cases) dimensions, where ϵ is a parameter that is set to zero at the end of the calculation. Following [18], the OPP is extended in $4+\epsilon$ dimensions with the introduction of rational terms. In the framework of the OPP, these rational terms are splitted in two, going by the name R_1 and R_2. The numerator is actually splitted in a four and an ϵ dimensional part. The R_1 piece comes from the four dimensional part of the numerator and its calculation can be performed in the same footing with the OPP method. However, the part arising from the ϵ part of the numerator cannot be treated in the same footing. As explained in chapter three, one can construct tree-level Feynman rules in the spirit of counterterms, by computing analytically the contribution of all one particle irreducible diagrams of a theory, up to four external legs. The fact that one needs diagrams up to four external legs is guaranteed by the ultraviolet nature of the rational piece. A tree-level calculation adds nothing in computing time in comparison with the one loop calculation that is needed for the rest of the amplitude. However, to calculate these Feynman rules analytically demands hard work since the number of irreducible diagrams up to four legs for specific theories can be very large. In [22] we gave the Feynman rules for the electroweak part of the standard model (of order of 10.000 Feynman diagrams were included) in the 't Hooft-Feynman gauge. Together with the works of [21] and [18] where the QCD and QED parts are given, we completed all the tools needed for any one loop calculation within the Standard Model. The results were given in the 't Hooft-Veltman and in the Four Dimensional Helicity schemes in chapter four and scheme dependence was discussed.

In case one wants to work in different gauges we repeated the calculation in general R_ξ and unitary gauges [23]. We presented our findings in chapter five, where we also discussed the gauge dependence of the R_2 and the R_1 pieces seperately. The whole rational part is gauge invariant.

Since the OPP method at one loop proved to be very succesful, we looked at possible extensions in two loops. There are similar attempts in the literature [43], [42]. The resulting base of integrals does not include scalar integrals anymore, there are also integrals with Irreducible Scalar Products in the numerator, raised to some power. Again, due to unitarity reasons, any integral in two loops is expected to be reduced to integrals up to 2d denominators in d dimensions. We extended our one loop reduction method to two loops. Again, by counting independent coefficients and tensor structures we proved that this is indeed the case, provided that the coefficients we assumed have up to cubic dependence in the loop momenta. There is an exception in 2 dimensions where due to some properties of the tensor structures, quadratic terms are sufficient. We presented our findings in chapter six for the case of two, three, four and five dimensions although we believe that this is a general property. Again, our method is not exactly the extension of the OPP method in two loops but as in the one loop case the two methods are connected. By using our findings one can rewrite the coefficients ‘a la OPP. The resulting base of integrals does respect unitarity and one can find the coefficients in the spirit of the OPP method.

However, we know from existing examples in the literature that in some cases, further reductions can be achieved and thus, the base we end up using only unitarity is not the minimal one. One has to use integration by parts identities, first introduced in [45]. We added total derivatives together with the rest pieces of our reduction in some examples and we were able to reproduce results known form the literature. In chapter seven we presented such examples. As a by-product, we can extend one loop OPP method in the case of integrals with denominators raised to some higher power.

Contrary to the one loop program, which is completely finished and automated in computer programs, there is still a lot of work to be done in this direction for the two loop cases. There are a lot of missing pieces. As difficult as this program may be, it is on the other hand exciting and necessary for a better understanding of our universe, especially at our times where colliders produce more and more data. So, independently of how hard one has to work in this field, the answers that we might get at the end are exciting and it definitely worths the trouble!

Samenvatting

Om fundamentele vragen in de wetenschap, zoals: waaruit bestaat ons universum, of waar komen de massa's van de elementaire deeltjes vandaan, bouwen natuurkundigen versnellers. Eén daarvan is de Large Hadron Collider (LHC), waarin protonen tot zeer hoge energieën worden versneld, om daarna met elkaar te botsen en andere deeltjes te maken, onder omstandigheden die die van de oerknal benaderen.

Wat daarop volgt lijkt op het zoeken naar spelden in hooibergen. De geproduceerde gegevens bevatten heel veel achtergrond-signalen, die moeten worden begrepen, en verwijderd, opdat meer interessante gebeurtenissen eruit kunnen springen, en nieuwe informatie kunnen leveren. Het moge duidelijk zijn dat hoe hoger de energieën, hoe groter de benodigde nauwkeurigheid waarmee deze achtergrondprocessen begrepen moeten worden. Zodra nieuwe deeltjes ontdekt worden, gaan deze op hun beurt achtergronden vormen bij weer nieuwe versnellers.

De manier waarop wij onze kennis en nauwkeurigheid vergroten is het berekenen van hogere-orde correcties in storingstheorie. Met behulp daarvan kunnen wij de zeer precieze theoretische resultaten vergelijken met de experimentele uitkomsten, en begrijpen wat de experimenten ons opleveren.

Dit betekent dat in de theoretische context Next-to-Leading-Order (NLO) en Next-to-Next-to-Leading-Order (NNLO) berekeningen met veel uitwendige deeltjes moeten worden betrokken. Een majeur gedeelte van deze NLO- en NNLO voorspellingen bestaat uit het berekenen van lus-correcties. In het geval van veel uitwendige deeltjes moet men zeer grote lus-integralen uitwerken, en dit geldt al vele jaren als de belangrijkste te nemen horde in zulke berekeningen.

De complexiteit van deze integralen maken, samen met het grote aantal relevante Feynman diagrammen, de berekeningen heel ingewikkeld. Een grote hulp hierbij zijn reductiemethoden, die het aantal uit te werken integralen kunnen verminderen. In het bijzonder reducties op het niveau van de integrand zelf [15], zoals bestudeerd in dit proefschrift, hebben de berekeningen aanzienlijk versneld omdat zij gemakkelijk in een computerprogramma kunnen worden geïmplementeerd, en in één keer de hele amplitude uitdrukken in een basis van Feynman-integralen, in plaats van dat ieder Feynman diagram afzonderlijk moet worden geëvalueerd.

Het doel van dit proefschrift is tweeërlei. Ten eerste is het het begrijpen van deze methodes op het één-lus niveau, en het verschaffen van de laatste ingrediënten die nodig zijn om het hele één-lus programma af te ronden. Ten tweede is het een begin te maken met het uitbreiden van deze methode naar het twee-lus niveau. In die richting ligt er nog heel wat werk in het verschiet, maar dat is nodig voor ons beter begrip van het universum. Er staat ons nog veel opwindends te wachten.

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Curriculum Vitae

The author of this thesis, Ioannis Malamos, was born and grew up in Athens, Greece. After attending high school, he studied physics in the University of Athens. After graduating from his bachelor, he continued in the Department of Nuclear Physics and High Energy Physics of the University of Athens, where he obtained his master's degree under the supervision of Professor Athanasios B. Lahanas. The title of his thesis was "Contributions of Higgs fields and gauge fields in the effective potential of the Minimal Supersymmetric Standard Model". After a short stay in NSCR Demokritos in Athens, to work with professor Costas G. Papadopoulos he moved in Nijmegen, The Netherlands to start his PhD project in Theoretical High Energy Physics with professor Ronald Kleiss and with copromotors professors Costas G. Papadopoulos and Roberto Pittau. His research in one and two loop calculations led to the writing of this thesis.